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Although in the first instance designed to supply a want, as a medium of communication between American mathematicians, its pages will always be open to contributions from abroad, and promises of support from various foreign mathmeticians of eminence have already been received.

The publication of original investigations is the primary object of the Journal. In addition to this, from time to time concise abstracts will be inserted of subjects to which special interest may attach, or which have been developed in memoirs difficult of access to American students.

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The editors believe it will materially aid in fostering the study of Mathematical Science throughout this continent and they feel it their duty to state that any good which may arise from it will be in a great measure due to the enlightened liberality of the Trustees of the Johns Hopkins University who have prompted the undertaking and guaranteed a considerable portion of the pecuniary risk attendant upon it.

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NOTE ON A CLASS OF TRANSFORMATIONS WHICH SURFACES MAY UNDERGO IN SPACE OF MORE THAN THREE DIMENSIONS.

By Simon Newcomb.

If the material bodies which surround us were placed in a space of more than three dimensions, their kinematic susceptibilities would be increased in a manner which, at first sight, would seem very extraordinary. Each body would, in fact, be susceptible of n independent forward motions, and $\frac{n(n-1)}{2}$ separate rotations, n being the number of dimensions of the space. My present purpose is not, however, to discuss the general theory of the subject, but to point out a special case of it as seen in a remarkable transformation to which closed surfaces may be subjected in space of four dimensions. The proposition in question may be expressed as follows:

If a fourth dimension were added to space, a closed material surface (or shell) could be turned inside out by simple flexure; without either stretching or tearing.

For simplicity we may suppose the surface to be spherical. Let

$$x, \ y, \ z, \ u,$$

be the general rectangular coördinates in the supposed space of four dimensions. An infinite plane space of three dimensions may then be represented by the equation

$$ax + by + cz + du = A,$$

a, b, etc., being any constants whatever. For simplicity we may suppose a, b and c all equal to zero, and the axes of x, y and z therefore to lie in the space of three dimensions under consideration. An Euclidian or natural space may then be represented by the single equation $u = A$, A being an arbitrary constant. The four-dimensional space may be divided into an infinity of Euclidian spaces by giving all possible values to A.

To define a surface in four-dimensional space by rectangular coördinates, two equations are necessary. If the surface is one which can exist in three-dimensional space, one of these equations must be of the first degree in x, y, etc. For example, the most general equations of the sphere in four-dimensional space are

$$(x-a)^2 + (y-b)^2 + (z-c)^2 + (u-d)^2 = r^2;$$

$$\alpha x + \beta y + \gamma z + \delta u = k, \tag{1}$$

α, β, γ, δ and k being constants, and a, b, c, d, the coördinates of the centre. This centre is not necessarily situated in the same Euclidian space with the surface; in fact there is a series of points, each of which is equidistant from every point of the surface, but only one of them lies in the same Euclidian space with the surface. This point is one whose coördinates fulfil the condition

$$\alpha a + \beta b + \gamma c + \delta d = k,$$

or,

$$\alpha (x-a) + \beta (y-b) + \gamma (z-c) + \delta (u-d) = 0.$$

If we choose our axes of coördinates so that the equations shall have the simplest forms, putting

$$\alpha = \beta = \gamma = 0, \quad \delta = 1,$$

the general equations (1) will become

$$(x-a)^2 + (y-b)^2 + (z-c)^2 + (u-d)^2 = r^2;$$

$$u = k.$$

Now, to consider the transformation of a material spherical surface, we must consider this surface as an indefinitely thin shell, situated between two surfaces. We shall suppose the natural space in which the sphere is situated in the beginning to be conditioned by the equation

$$u = k = 0,$$

and we shall take the centre of the sphere as the origin of coördinates. The equations of the inner surface of the spherical shell, which we may call A, will then be of the form

$$x^2 + y^2 + z^2 = r^2; \quad u = 0,$$

and of the outer one, B

$$x_1^2 + y_1^2 + z_1^2 = r_1^2; \quad u = 0.$$

Now suppose that, the outer surface remaining fixed, we move the inner one in the direction of the axis of u by a small quantity k, allowing its radius

at the same time to vary in such a way that the thickness of the shell shall remain unchanged. Its equations may then be expressed in the form

$$x^2 + y^2 + z^2 + u^2 = (r + \delta r)^2\,;\ u = k,$$

and we must, if possible, determine δr so that the thickness of the shell shall remain unaltered. To find the new thickness, we remark that the square of the distance of any point of the outer from any point of the inner surface is

(2) $$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 + u^2 .$$

The condition that this shall be a minimum for a given point of the outer surface is

$$(x - x_1)\,dx + (y - y_1)\,dy + (z - z_1)\,dz = 0,$$

dx, dy and dz being subject to the condition

$$xdx + ydy + zdz = 0.$$

The simultaneous existence of these equations depends upon our having

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1}\,,$$

or,

(3) $$x = \lambda x_1\,,\quad y = \lambda y_1\,,\quad z = \lambda z_1\,,$$

λ being a quantity which differs very little from unity. These values of x, y and z, being substituted in (2), give for the square of the thickness of the shell after the motion of the inner surface

$$(\lambda - 1)^2\,(x_1^2 + y_1^2 + z_1^2) + k^2 = (\lambda - 1)^2\, r_1^2 + k^2 .$$

Let us put $h = r_1 - r$, for the original thickness of the shell. In order that the thickness may remain unaltered, it is necessary and sufficient that we determine k and λ simultaneously in terms of an arbitrary angle θ by the conditions

(4) $$k = h \sin\theta\,;\ 1 - \lambda = \frac{h}{r_1}\cos\theta.$$

The original position of the inner surface will be that corresponding to $\theta = 0$. Suppose, now, that θ increases from 0° to 180°, k and λ being constantly determined by the condition (4). It is then evident that the shell will experience no other deformation than that arising from flexure, the flexure involving a stretching of the outer surface which may be made indefinitely small by diminishing the thickness of the shell. When θ reaches 180° we shall once more have $k = 0$, so that the surface A will be brought back into

its original natural or Euclidian space. Moreover, λ being then greater than unity, the equations (3) show that the radius of this surface A, or $x^2 + y^2 + z^2$, will be greater than that of B.

The outer surface $x_1^2 + y_1^2 + z_1^2 = r_1^2$ will therefore be the inner one, and the other the outer one, the change being brought about by flexure alone. By the motion we have supposed, every point of the surface A has described a semicircle round the corresponding point of B, θ representing the angle of position of the line joining the two points during the motion. For simplicity we have supposed the surface A only to vary, the flexure taking place round B, but we might equally have supposed a mean surface to remain constant.

RESEARCHES IN THE LUNAR THEORY,*

By G. W. Hill, *Nyack Turnpike, N. Y.*

When we consider how we may best contribute to the advancement of this much-treated subject, we cannot fail to notice that the great majority of writers on it have had before them, as their ultimate aim, the construction of Tables: that is they have viewed the problem from the stand-point of practical astronomy rather than of mathematics. It is on this account that we find such a restricted choice of variables to express the position of the moon, and of parameters, in terms of which to express the coëfficients of the periodic terms. Again, their object compelling them to go over the whole field, they have neglected to notice many minor points of great interest to the mathematician, simply because the knowledge of them was unnecessary for the formation of Tables. But the developments having now been carried extremely far, without completely satisfying all desires, one is led to ask whether such modifications cannot be made in the processes of integration, and such coördinates and parameters adopted, that a much nearer approach may be had to the law of the series, and, at the same time, their convergence augmented.

Now, as to choice of coördinates, it is known that, in the elliptic theory, the rectangular coördinates of a planet, relative to the central body, the axes being parallel to the axes of the ellipse described, can be developed, in terms of the time, in the following series,

$$x = a \sum_{i=-\infty}^{i=+\infty} \frac{1}{i} J_{\frac{ie}{2}}^{(i-1)} \cos ig,$$

$$y = b \sum_{i=-\infty}^{i=+\infty} \frac{1}{i} J_{\frac{ie}{2}}^{(i-1)} \sin ig,$$

a and b being the semi-axes of the ellipse, e the eccentricity, g the mean anomaly and, for positive values of i, the Besselian function (in Hansen's notation)

$$J_{\lambda}^{(i)} = \frac{\lambda^i}{1.2\ldots i}\left[1 - \frac{\lambda^2}{1.(i+1)} + \frac{\lambda^4}{1.2.(i+1)(i+2)} - \ldots\right],$$

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* Communicated to the National Academy of Sciences at the session of April, 1877.

while, for negative values of i, we have

$$J_{\lambda}^{(-i)} = J_{-\lambda}^{(i)},$$

and, for the special case of $i = 0$, we put the indeterminate

$$\tfrac{1}{0} J_{0}^{(-1)} = -\tfrac{3}{2} e.$$

Here the law of the series is manifest, and the approximation can easily be carried as far as we wish. But the longitude and latitude, variables employed by nearly all the lunar theorists, are far from having such simple expressions; in fact, their coëfficients cannot be expressed finitely in terms of Besselian functions. And if this is true in the elliptic theory, how much more likely is a similar thing to be true when the complexity of the problem is increased by the consideration of disturbing forces? We are then justified in thinking that the coëfficients of the periodic terms in the development of rectangular or quasi-rectangular coördinates are less complex functions of their parameters than those of polar coördinates. There is also another advantage in employing coördinates of the former kind; the differential equations are expressed in purely algebraic forms; while, with the latter, circular functions immediately present themselves. For these reasons I have not hesitated to substitute rectangular for polar coördinates.

Again, as to parameters, all those who have given literal developments, Laplace setting the example, have used the parameter m, the ratio of the sidereal month to the sidereal year. But a slight examination, even, of the results obtained, ought to convince any one that this is a most unfortunate selection in regard to convergence. Yet nothing seems to render this parameter at all desirable, indeed, the ratio of the synodic month to the sidereal year would appear to be more naturally suggested as a parameter. Some instances of slow convergence with the parameter m may be given from Delaunay's Lunar Theory: the development of the principal part of the coëfficient of the evection in longitude begins with the term $\frac{15}{4} me$, and ends with the term $\frac{413277465931033}{15288238080} m^8 e$; again, in the principal part of the coëfficient of the inequality whose argument is the difference of the mean anomalies of the sun and moon, we find, at the beginning, the term $\frac{21}{4} mee'$, and, at the end, the term $\frac{1207454026843}{3538944} m^7 ee'$. It is probable that, by the adoption of some function of m as a parameter in place of this quantity, whose numerical

value, in the case of our moon, should not greatly exceed that of m, the foregoing large numerical coëfficients might be very much diminished. And nothing compels us to use the same parameter throughout; one may be used in one class of inequalities, another in another, as may prove most advantageous. It is known what rapid convergence has been obtained in the series giving the values of logarithms, circular and elliptic functions, by simply adopting new parameters. Similar transformations, with like effects, are, perhaps, possible in the coëfficients of the lunar inequalities. However, as far as my experience goes, no useful results are obtained by experimenting with the present known developments; in every case it seems the proper parameter must be deduced from *a priori* considerations furnished in the course of the integration.

With regard to the form of the differential equations to be employed, although Delaunay's method is very elegant, and, perhaps, as short as any, when one wishes to go over the whole ground of the lunar theory, it is subject to some disadvantages when the attention is restricted to a certain class of lunar inequalites. Thus, do we wish to get only the inequalities whose coëfficients depend solely on m, we are yet compelled to develop the disturbing function R to all powers of e. Again, the method of integrating by undetermined coëfficients is most likely to give us the nearest approach to the law of the series; and, in this method, it is as easy to integrate a differential equation of the second order as one of the first, while the labor is increased by augmenting the number of variables and equations. But Delaunay's method doubles the number of variables in order that the differential equations may be all of the first order. Hence, in this disquisition, I have preferred to use the equations expressed in terms of the coördinates, rather than those in terms of the elements; and, in general, always to diminish the number of unknown quantities and equations by augmenting the order of the latter. In this way the labor of making a preliminary development of R in terms of the elliptic elements is avoided.

In the present memoir I propose, dividing the periodic developments of the lunar coördinates into classes of terms, after the manner of Euler in his last Lunar Theory,* to treat the following five classes of inequalities:—

1. Those which depend only on the ratio of the mean motions of the sun and moon.

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* *Theoria Motuum Lunæ, nova methodo pertractata. Petropoli, 1772.*

2. Those which are proportional to the lunar eccentricity.
3. Those which are proportional to the sine of the lunar inclination.
4. Those which are proportional to the solar eccentricity.
5. Those which are proportional to the solar parallax.

A general method will also be given by which these investigations may be extended so as to cover the whole ground of the lunar theory. My methods have the advantage, which is not possessed by Delaunay's, that they adapt themselves equally to a special numerical computation of the coëfficients, or to a general literal development. The application of both procedures will be given in each of the just mentioned five classes of inequalities, so that it will be possible to compare them.

I regret that, on account of the difficulty of the subject and the length of the investigations it seems to require, I have been obliged to pass over the important questions of the limits between which the series are convergent, and of the determination of superior limits to the errors committed in stopping short at definite points. There cannot be a reasonable doubt that, in all cases, where we are compelled to employ infinite series in the solution of a problem, analysis is capable of being perfected to the point of showing us within what limits our solution is legitimate, and also of giving us a limit which its error cannot surpass. When the coördinates are developed in ascending powers of the time, or in ascending powers of a parameter attached as a multiplier to the disturbing forces, certain investigations of Cauchy afford us the means of replying to these questions. But when, for powers of the time, are substituted circular functions of it, and the coëfficients of these are expanded in powers and products of certain parameters produced from the combination of the masses with certain of the arbitrary constants introduced by integration, it does not appear that anything in the writings of Cauchy will help us to the conditions of convergence.

CHAPTER I.

Differential Equations.—Properties of motion derived from Jacobi's integral.

We set aside the action of the planets and the influence of the figures of the sun, earth and moon, together with the action of the last upon the

sun, as also the product of the solar disturbing force on the moon by the small fraction obtained from dividing the mass of the earth by the mass of the sun. These are the same restrictions as those which Delaunay has imposed on his Lunar Theory contained in Vols. XXVIII and XXIX of the Memoirs of the Paris Academy of Sciences. The motion of the sun, about the earth, is then in accordance with the elliptic theory, and the ecliptic is a fixed plane.

Let us take a system of rectangular axes, having its origin at the centre of gravity of the earth, the axis of x being constantly directed toward the centre of the sun, the axis of y toward a point in the ecliptic 90° in advance of the sun, and the axis of z perpendicular to the ecliptic. In addition, we adopt the following notation: —

$r' =$ the distance of the sun from the earth;

$\lambda' =$ the sun's longitude;

$\mu =$ the sum of the masses of the earth and moon, measured by the velocity these masses produce by their action, in a unit of time, and at the unit of distance;

$m' =$ the mass of the sun, measured in the same way;

$n' =$ the mean angular velocity of the sun about the earth;

$a' =$ the sun's mean distance from the earth.

In accordance with one of the above-mentioned restrictions we have the equation: —

$$m' = n'^2 a'^3$$

The axes of x and y having a velocity of rotation in their plane, equal to $\frac{d\lambda'}{dt}$, it is evident that the square of the velocity of the moon, relative to the earth's centre, has for expression, in terms of the adopted coördinates,

$$\begin{aligned} 2\,T &= \left[\frac{dx}{dt} - y\frac{d\lambda'}{dt}\right]^2 + \left[\frac{dy}{dt} + x\frac{d\lambda'}{dt}\right]^2 + \frac{dz^2}{dt^2} \\ &= \frac{dx^2 + dy^2 + dz^2}{dt^2} + 2\,\frac{d\lambda'}{dt}\,\frac{xdy - ydx}{dt} + \frac{d\lambda'^2}{dt^2}\,(x^2 + y^2). \end{aligned}$$

The potential function, in terms of the same coördinates, is

$$\Omega = \frac{\mu}{\sqrt{(x^2 + y^2 + z^2)}} + \frac{n'^2\,a'^3}{\sqrt{[(r' - x)^2 + y^2 + z^2]}} - \frac{n'^2\,a'^3}{r'^2}\,x.$$

If the second radical in this expression is expanded in a series proceeding according to descending powers of r', and the first term $\frac{n'^2 a'^3}{r'}$ omitted, since it

disappears in all differentiations with respect to the moon's coördinates, the following expression is obtained

$$\Omega = \frac{\mu}{\sqrt{(x^2+y^2+z^2)}} + n'^2 \frac{a'^3}{r'^3}\left[x^2 - \tfrac{1}{2}(y^2+z^2)\right]$$
$$+ \frac{n'^2}{a'}\frac{a'^4}{r'^4}\left[x^3 - \tfrac{3}{2}x(y^2+z^2)\right]$$
$$+ \frac{n'^2}{a'^2}\frac{a'^5}{r'^5}\left[x^4 - 3x^2(y^2+z^2) + \tfrac{3}{8}(y^2+z^2)^2\right]$$
$$+ \frac{n'^2}{a'^3}\frac{a'^6}{r'^6}\left[x^5 - 5x^3(y^2+z^2) + \tfrac{15}{8}x(y^2+z^2)^2\right]$$
$$+ \cdots\cdots\cdots\cdots$$

Since the differential equations of motion are of the form

$$\frac{d}{dt}\cdot\frac{dT}{d.\frac{d\phi}{dt}} - \frac{dT}{d\phi} = \frac{d\Omega}{d\phi},$$

ϕ denoting, in succession, each of the variables which define the position of the moon, it is plain that the term

$$\tfrac{1}{2}\frac{d\lambda'^2}{dt^2}(x^2+y^2)$$

may be removed from T and added to Ω; and these modified quantities may be denoted by the symbols T' and Ω'. Then these equations may be written thus:

$$\frac{d^2x}{dt^2} - 2\frac{d\lambda'}{dt}\frac{dy}{dt} - \frac{d^2\lambda'}{dt^2}y = \frac{d\Omega'}{dx},$$
$$\frac{d^2y}{dt^2} + 2\frac{d\lambda'}{dt}\frac{dx}{dt} + \frac{d^2\lambda'}{dt^2}x = \frac{d\Omega'}{dy},$$
$$\frac{d^2z}{dt^2} = \frac{d\Omega'}{dz}.$$

When we wish to restrict our attention to the lunar inequalities which are independent of the solar parallax, all the terms, in the last expression of Ω, which are divided by r'^4, r'^5, r'^6, &c., may be omitted. In this case it will be seen that all the terms, introduced into the differential equations by the solar action, are linear in form, with variable, but known coëfficients, since $\frac{d\lambda'}{dt}$, $\frac{d^2\lambda'}{dt^2}$ and $\frac{a'^3}{r'^3}$ are known functions of t.

When all the inequalities, involving the solar eccentricity, are neglected, the equations admit an integral in finite terms. For, in this case, we have

$$\frac{d\lambda'}{dt} = n', \qquad \frac{d^2\lambda'}{dt^2} = 0, \qquad r' = a',$$

and Ω' does not explicitly contain t; hence, multiplying the equations severally by the factors dx, dy, and dz, and adding the products, both members of the resulting equation are exact differentials. Integrating this equation, we have

$$\frac{dx^2 + dy^2 + dz^2}{2dt^2} = \Omega' + \text{a constant}.$$

This integral equation appears to have been first obtained by Jacobi.* As it will be frequently referred to in what follows, I shall take the liberty of calling it Jacobi's integral.

If we take two imaginary variables

$$u = x + \sqrt{(y^2 + z^2)}\sqrt{-1},$$
$$s = x - \sqrt{(y^2 + z^2)}\sqrt{-1},$$

Ω has the following simple expression, being a function of two variables only,

$$\Omega = \frac{\mu}{\sqrt{us}} + \frac{n'^2 a'^3}{\sqrt{(r' - u)}\sqrt{(r' - s)}} - \frac{n'^2 a'^3}{2r'^2}(u + s).$$

If this is expanded in descending powers of r', and, as before, the term $\frac{n'^2 a^3}{r'}$ omitted,

$$\Omega = \frac{\mu}{\sqrt{us}} + n'^2 \frac{a'^3}{r'^3}\left[\tfrac{3}{8} u^2 + \tfrac{1}{4} us + \tfrac{3}{8} s^2\right]$$
$$+ \frac{n'^2}{a'}\frac{a'^4}{r'^4}\left[\tfrac{5}{16} u^3 + \tfrac{3}{16} u^2 s + \tfrac{3}{16} us^2 + \tfrac{5}{16} s^3\right]$$
$$+ \frac{n'^2}{a'^2}\frac{a'^5}{r'^5}\left[\tfrac{35}{128} u^4 + \tfrac{5}{32} u^3 s + \tfrac{9}{64} u^2 s^2 + \tfrac{5}{32} us^3 + \tfrac{35}{128} s^4\right]$$
$$+ \ \ldots\ldots\ldots\ldots$$

The additional variable, necessary to complete the definition of the moon's position, may be so taken that the expression of T may be simplified as much as possible. This expression may be written

$$2\,T = \frac{duds}{dt^2} - 4\frac{(ydz - zdy)^2}{(u-s)^2 dt^2} + 2\frac{d\lambda'}{dt}\,\frac{xdy - ydx}{dt} + \frac{d\lambda'^2}{dt^2}(us - z^2)\,.$$

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* *Comptes Rendus de l'Académie des Sciences de Paris.* Tom. iii, p. 59.

There does not seem to be any function of x, y and z, which, adopted as a new variable to accompany u and s, would reduce this to a very simple form. However, when we are engaged in determining the inequalities independent of the inclination of the lunar orbit, this transformation will be useful to us. For, in this case, $z = 0$, and the values of u and s become

$$u = x + y\sqrt{-1},$$
$$s = x - y\sqrt{-1},$$

and T is given by the equation

$$2\,T = \frac{duds}{dt^2} - \frac{d\lambda'}{dt}\,\frac{uds - sdu}{dt} + \frac{d\lambda'^2}{dt^2}\,us.$$

Although Ω is expressed most simply by the systems of coördinates we have just employed, the integration of the differential equations will be easier, if we suppose that the axes of x and y have a constant instead of a variable velocity of rotation, the axis of x being made to pass through the mean position of the sun instead of the true. To obtain the expression for T correspondent to this supposition, it is necessary only to write n' for $\frac{d\lambda'}{dt}$ in the former values. As for Ω, it can be written thus

$$\Omega = \frac{\mu}{r} + \frac{n'^2 a'^3}{[r'^2 - 2r'S + r^2]^{\frac{1}{2}}} - \frac{n'^2 a'^3}{r'^2}\,S,$$

where

$r^2 = x^2 + y^2 + z^2 =$ the square of the moon's radius vector;

$S = x\cos v + y\sin v$;

$v =$ the solar equation of the centre.

This function being expanded in a series of descending powers of r', as before, we have

$$\begin{aligned}\Omega' = {} & \frac{\mu}{r} + \tfrac{1}{2}\,n'^2\,(x^2 + y^2) \\ & + n'^2\,\frac{a'^3}{r'^3}\left[\tfrac{3}{2}\,S^2 - \tfrac{1}{2}\,r^2\right] \\ & + \frac{n'^2}{a'}\,\frac{a'^4}{r'^4}\left[\tfrac{5}{2}\,S^3 - \tfrac{3}{2}\,r^2 S\right] \\ & + \frac{n'^2}{a'^2}\,\frac{a'^5}{r'^5}\left[\tfrac{35}{8}\,S^4 - \tfrac{15}{4}\,r^2 S^2 + \tfrac{3}{8}\,r^4\right] \\ & + \frac{n'^2}{a'^3}\,\frac{a'^6}{r'^6}\left[\tfrac{63}{8}\,S^5 - \tfrac{35}{4}\,r^2 S^3 + \tfrac{15}{8}\,r^4 S\right] \\ & + \ldots\ldots\ldots\ldots\end{aligned}$$

And the corresponding differential equations are

$$\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} = \frac{d\Omega'}{dx},$$

$$\frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} = \frac{d\Omega'}{dy},$$

$$\frac{d^2z}{dt^2} = \frac{d\Omega'}{dz},$$

Thus much in reference to the equations under as general a form as we shall have occasion for in the present disquisition. We shall now suppose that they are reduced to as restricted a form as is possible without their becoming the equations of the elliptic theory; that is, we shall assume that the solar parallax and eccentricity and the lunar inclination vanish. With these simplifications, in the first system of coördinates,

$$T' = \frac{dx^2+dy^2}{2dt^2} + n'\frac{xdy-ydx}{dt},$$

$$\Omega' = \frac{\mu}{\sqrt{(x^2+y^2)}} + \tfrac{3}{2}n'^2x^2;$$

and, in the second,

$$T' = \frac{duds}{2dt^2} - \frac{n'}{2}\frac{uds-sdu}{dt},$$

$$\Omega' = \frac{\mu}{\sqrt{us}} + \tfrac{3}{8}n'^2(u+s)^2.$$

And the differential equations, correspondent, are, in the first case,

$$\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} + \left[\frac{\mu}{r^3} - 3n'^2\right]x = 0,$$

$$\frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} + \frac{\mu}{r^3}y = 0,$$

and, in the second,

$$\frac{d^2u}{dt^2} - 2n'\frac{du}{dt} + \frac{\mu}{(us)^{\frac{3}{2}}}u - \tfrac{3}{2}n'^2(u+s) = 0,$$

$$\frac{d^2s}{dt^2} + 2n'\frac{ds}{dt} + \frac{\mu}{(us)^{\frac{3}{2}}}s - \tfrac{3}{2}n'^2(u+s) = 0.$$

The Jacobian integral has severally the expressions

$$\frac{dx^2+dy^2}{2dt^2} = \frac{\mu}{r} + \tfrac{3}{2}n'^2x^2 - C,$$

$$\frac{duds}{2dt^2} = \frac{\mu}{\sqrt{us}} + \tfrac{3}{8}n'^2(u+s)^2 - C.$$

The terms $-2n'\frac{dy}{dt}$, $2n'\frac{dx}{dt}$, &c., have been introduced into the equations by making the axes of coördinates movable; but since the putting of $n' = 0$ makes the solar disturbing force vanish, there is no inconsistency in attributing them to the solar action. Then, in the case of the vanishing of this action, we have the equations of ordinary elliptic motion

$$\frac{d^2x}{dt^2} + \frac{\mu}{r^3}x = 0,$$

$$\frac{d^2y}{dt^2} + \frac{\mu}{r^3}y = 0.$$

Thus, in the restricted case we consider, all the terms, added to the differential equations of motion by the solar action, are linear in form and have constant coëfficients. This, and the circumstance that t does not explicitly appear in the equations, are two advantages which are due to the particular selection of the variables x and y. If $\frac{\mu}{r^3}$ were constant, the equations would be linear with constant coëfficients and easily integrable.

The constants μ and n' can be made to disappear from the differential equations, if, instead of leaving the units of length and time arbitrary, we assume them so that $\mu = 1$, and $n' = 1$. The new unit of length, will then be equal to $\sqrt[3]{\frac{\mu}{n'^2}}$ units of the previous measurement. The equations, thus simplified, are

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} + \left[\frac{1}{r^3} - 3\right]x = 0,$$

$$\frac{d^2y}{dt^2} + 2\frac{dx}{dt} + \frac{1}{r^3}y \qquad = 0,$$

with their integral

$$\frac{dx^2 + dy^2}{dt^2} = \frac{2}{r} + 3x^2 - 2C.$$

It will be perceived that, in this way, we make the differential equations absolutely the same for all cases of the satellite problem.

Let us put $\rho = \frac{dx}{dt}$, then

$$\frac{dy}{dt} = \left[\frac{2}{r} + 3x^2 - 2C - \rho^2\right]^{\frac{1}{2}},$$

$$\frac{d\rho}{dt} = 2\left[\frac{2}{r} + 3x^2 - 2C - \rho^2\right]^{\frac{1}{2}} - \left[\frac{1}{r^3} - 3\right]x.$$

Or, by making y the independent variable,

$$\frac{dx}{dy} = \frac{\rho}{\left[\frac{2}{r} + 3x^2 - 2C - \rho^2\right]^{\frac{1}{2}}},$$

$$\frac{d\rho}{dy} = 2 - \frac{\left[\frac{1}{r^3} - 3\right] x}{\left[\frac{2}{r} + 3x^2 - 2C - \rho^2\right]^{\frac{1}{2}}}.$$

The problem is then reduced to the integration of two differential equations of the first order. Were this accomplished, and ρ eliminated from the two integral equations, we should have the equation of the orbit. If we put

$$W = 2x + \left[\frac{2}{r} + 3x^2 - 2C - \rho^2\right]^{\frac{1}{2}},$$

the differential equations can be written in the canonical form,

$$\frac{dx}{dy} = - \frac{dW}{d\rho},$$

$$\frac{d\rho}{dy} = \frac{dW}{dx}.$$

It may be worth while to notice also the single partial differential equation, to the integration of which our problem can be reduced. Returning to the arbitrary linear and temporal units, and, for convenience, reversing the sign of C, if a function of x and y can be found satisfying the partial differential equation

$$\left[\frac{dV}{dx} + n'y\right]^2 + \left[\frac{dV}{dy} - n'x\right]^2 = \frac{2\mu}{\sqrt{(x^2 + y^2)}} + 3n'^2x^2 + 2C,$$

and involving a single arbitrary constant h, distinct from that which can be joined to it by addition, the intermediate integrals of the problem will be

$$\frac{dx}{dt} = \frac{dV}{dx} + n'y, \qquad \frac{dy}{dt} = \frac{dV}{dy} - n'x,$$

and the final integrals

$$\frac{dV}{dh} = \alpha, \qquad \frac{dV}{dC} = t + c,$$

α and c being two additional arbitrary constants. The truth of this will be evident if we differentiate the four integral equations with respect to t and compare severally the results with the partial differential coëfficients of the partial differential equation with respect to x, y, h and C.

Although, in this manner, the problem seems reduced to its briefest terms, yet, when we essay to solve it, setting out with this partial differential equation, we are led to more complex expressions than would be expected. It would be advisable, in this method of proceeding, to substitute polar for rectangular coördinates, or to put

$$x = r\cos\phi, \qquad y = r\sin\phi.$$

The partial differential equation, thus transformed, is

$$\frac{dV^2}{dr^2} + \left[\frac{1}{r}\frac{dV}{d\phi} - n'r\right]^2 = \frac{2\mu}{r} + \tfrac{3}{2}n'^2r^2 + 2C + \tfrac{3}{2}n'^2r^2\cos 2\phi.$$

This would have to be integrated by successive approximations, and it is found that this method, which, at first sight, seems likely to afford a briefer solution of the problem, because but one unknown quantity was to be determined, and this free from the variable t, and involving only half of the number of arbitrary constants introduced by integration, when developed, leads to as complex operations as the older methods, and, moreover, has the disadvantage of giving results which need prolix transformations before the coördinates can be exhibited in terms of the time.

Although we shall make no use of the equations in terms of polar coördinates, they may be given here, for the sake of some special properties they possess in this form. They are

$$r\frac{d^2r}{dt^2} - r^2\frac{d\phi^2}{dt^2} - 2n'r^2\frac{d\phi}{dt} + \frac{\mu}{r} - 3n'^2r^2\cos^2\phi = 0,$$

$$\frac{d}{dt}\left[r^2\left(\frac{d\phi}{dt} + n'\right)\right] + \tfrac{3}{2}n'^2r^2\sin 2\phi \qquad = 0,$$

with their integral

$$\frac{dr^2 + r^2d\phi^2}{dt^2} = \frac{2\mu}{r} + 3n'^2r^2\cos^2\phi - 2C.$$

By dividing the second of the differential equations by r^2, the variables are separated, and λ denoting the longitude of the moon, we have

$$r = \frac{K}{\sqrt{\frac{d\lambda}{dt}}}\,\varepsilon^{-\frac{3}{4}n'^2\int\frac{\sin 2(\lambda-\lambda')}{\frac{d\lambda}{dt}}dt},$$

K being a constant. Thus, after the longitude is determined in terms of t, the radius vector is obtained by a quadrature. But it can also be found, without the necessity of an integration, by dividing the integral by r^2 and

then eliminating the term $\frac{1}{r^2}\frac{dr^2}{dt^2}$ by means of its value derived from the second differential equation; in this way we get

$$\frac{\mu}{r^3}-\frac{C}{r^2}=\tfrac{1}{8}\left[\frac{\frac{d^2\phi}{dt^2}+\frac{3}{2}n'^2\sin 2\phi}{\frac{d\phi}{dt}+n'}\right]^2+\tfrac{1}{2}\frac{d\phi^2}{dt^2}-\tfrac{3}{2}n'^2\cos^2\phi.$$

As we desire to make constant numerical application of the general theory, established in what follows, to the particular case of the moon, we delay here, for a moment, to obtain the numerical values of the three constants μ, n' and C. The value of μ may be derived either from the observed value of the constant of lunar parallax combined with the mean angular motion of the moon, or from the intensity of gravity at the earth's surface and the ratio of the moon's mass to that of the earth. We will adopt the latter procedure. The value of gravity at the equator, $g = 9.779741$ metres, the unit of time being the mean solar second. We propose, however, to take the mean solar day as the unit of time, and the equatorial radius of the earth as the linear unit. This number must then be multiplied by $\frac{86400^2}{6377397.15}$, (6377397.15 metres is Bessel's value of the equatorial radius.) Moreover, the theory of the earth's figure shows that, in order to obtain the attractive force of the earth's mass, considered as concentrated at its centre of gravity, a second multiplication must be made by the factor 1.001818356. With our units then this force is represented by the number 11468.338: and the moon's mass being taken at $\frac{1}{81.52277}$ of the earth's, her attractive force is represented by the number 140.676. Consequently

$$\mu = 11609.014.$$

The sidereal mean motion of the sun in a Julian year is $1295977''.41516$, whence

$$n' = 0.017202124.$$

The value of C might be obtained from the observed values of the moon's coördinates and their rates of variation at any time. However, as the eccentricity of the earth's orbit is not zero, C obtained in this manner would be found to undergo slight variations. The mean of all the values obtained in a long series of observations might be adopted as the proper value of this

quantity when regarded as constant. But it is much easier to derive it approximately from the series

$$2C = (\mu n)^{\frac{2}{3}} \left[1 + 2\mathrm{m} - \tfrac{3}{2}\,\mathrm{m}^2 - \mathrm{m}^3 - \tfrac{1319}{288}\,\mathrm{m}^4 - \tfrac{67}{144}\,\mathrm{m}^5 - \tfrac{2879}{1296}\,\mathrm{m}^6 - \tfrac{1321}{1296}\,\mathrm{m}^7\right],$$

which will be established in the following chapter. Here n denotes the moon's sidereal mean motion, and m is put for $\frac{n'}{n - n'}$. In this formula the terms which involve the squares of the lunar eccentricity and inclination and of the solar parallax are neglected; this, however, is not of great moment, as, being multipled by at least m^2, they are of the fourth order with respect to smallness. The observations give $n = 0.22997085$, hence

$$C = 111.18883.$$

If it is proposed to assume the units of time and length so that μ and n' may both be unity, it will be found that the first is equal to 58.13236 mean solar days, and the second to 339.7898 equatorial radii of the earth. The corresponding value of C is 3.254440.

Let us now notice some of the properties of motion which can be derived from Jacobi's integral. This integral gives the square of the velocity relative to the moving axes of coördinates; and, as this square is necessarily positive, the putting it equal to zero gives the equation of the surface which separates those portions of space, in which the velocity is real, from those in which it is imaginary. This equation is, in its most general form,

$$\frac{\mu}{\sqrt{(x^2 + y^2 + z^2)}} + \frac{n'^2 a'^3}{\sqrt{[(a' - x)^2 + y^2 + z^2]}} = C + \tfrac{3}{2} n'^2 a'^2 - \frac{n'^2}{2} [(a' - x)^2 + y^2],$$

which is seen to be of the 16th degree. As y and z enter it only in even powers, the surface is symmetrically situated with respect to the planes of xy and xz. The left member is necessarily positive, (the folds of the surface, for which either or both the radicals receive negative values, are excluded from consideration), hence the surface is inclosed within the cylinder whose axis passes through the centre of the sun perpendicularly to the ecliptic, and whose trace on this plane is a circle of the radius

$$a' \sqrt{\left(3 + \frac{2C}{n'^2 a'^2}\right)}.$$

As, in general, the second term of the quantity, under the radical sign, is much smaller than the first, this radius is, quite approximately $\sqrt{3}\, a'$. Thus, in the case of our moon, assuming $\frac{1}{a'} = \sin 8''.848$, we have this radius $=$

$\sqrt{3.001383}\,a'$. It is evident that, for all points without this cylinder, the velocity is real; and as, for large values of z, whether positive or negative, the left member of the equation becomes very small, it is plain that the cylinder is asymptotic to the surface. Every right line, perpendicular to the ecliptic, intersects the surface not more than twice, at equal distances from this plane, once above and once below.

Let us now find the trace of the surface on the plane of xy. Putting ρ for the distance of a point on this trace from the centre of the sun,

$$\rho^2 = (a' - x)^2 + y^2.$$

and it is evident that the cubic equation,

$$\rho^3 = a'^2\left(3 + \frac{2C}{n'^2a'^2}\right)\rho - 2a'^3,$$

will give the limits between which the values of ρ can oscillate. If C is negative, this equation has but one real root which is negative; consequently, in this case, the surface has no intersection with the plane of xy. But, in all the satellite systems we know, C is positive, and this condition is probably necessary to insure stability. Hence we shall restrict our attention to the case where C is positive. Then all the roots of the equation are real, and two are positive. It is between the latter roots that ρ must always be found. To compute them, we derive the auxillary angle θ from the formula

$$\sin\theta = \left[1 + \tfrac{2}{3}\frac{C}{n'^2a'^2}\right]^{-\frac{3}{2}},$$

or, since θ differs but little from $90°$, with more readiness from

$$\cos^2\theta = \frac{2\dfrac{C}{n'^2a'^2}\left[1 + \tfrac{2}{3}\dfrac{C}{n'^2a'^2} + \tfrac{4}{27}\dfrac{C^2}{n'^4a'^4}\right]}{\left[1 + \tfrac{2}{3}\dfrac{C}{n'^2a'^2}\right]^3},$$

or, as $\dfrac{C}{n'^2a'^2}$ is a small quantity, with sufficient approximation from

$$\cos\theta = \frac{\sqrt{2\dfrac{C}{n'^2a'^2}}}{1 + \tfrac{2}{3}\dfrac{C}{n'^2a'^2}}.$$

The two roots are then

$$\rho_1 = 2a'\sqrt{\left(1 + \tfrac{2}{3}\frac{C}{n'^2a'^2}\right)}\sin\frac{\theta}{3},$$

$$\rho_2 = 2a'\sqrt{\left(1 + \tfrac{2}{3}\frac{C}{n'^2a'^2}\right)}\sin\left(60° - \frac{\theta}{3}\right).$$

The trace of the surface on the plane of xy is then wholly comprised in the annular space between the two circles described from the centre of the sun as centre with the radii ρ_1 and ρ_2. Moreover, as in most satellite systems we have $\frac{\mu}{n'^2a'^3}$ equal to a very small fraction, $\left(\text{for our moon } \frac{\mu}{n'^2a'^3} = \frac{1}{322930\ 2}\right)$, it is plain that, for points whose distance from the earth is comparable with their distance from the sun, the trace is approximately coïncident with these circles. For the term $\frac{\mu}{r}$, in the equation, may then be neglected in comparison with the other terms.

In the case of our moon there is found

$$\theta = 87^\circ\ 52'\ 11''.53.$$

and hence

$$\rho_1 = 22815.15. \qquad \rho_2 = 23816.09,$$

and, if r and ρ are regarded as the variables defining the position of a point in the plane xy, the following table gives some corresponding values of these quantities, for each of the two branches of the trace approximating severally to the two circles.

r.	ρ.	r.	ρ.
433.3257	22878 69	439.7922	23751.81
450	22876.17	450	23753.37
500	22869.68	500	23760.04
600	22860.13	600	23769.85
1000	22841.59	1000	23788.87
10000	22817.70	10000	23813.43
46127.70	22815.68	47127.55	23815.53

The first and last values correspond to the four points where the curves intersect the axis of x on the hither and thither side of the sun. It will be seen that the approximation of the branches to the circles is quite close, except in the vicinity of the earth, where there is a slight protruding away from them.

In addition to these two branches of the trace, there is, in the case where C exceeds a certain limit, a third closed one about the origin much smaller than the former. As the coördinates of points in this branch are small fractions of a', its equation may be written, quite approximately,

$$\frac{\mu}{r} = C - \tfrac{3}{2}\, n'^2x^2.$$

It intersects the axis of y at a distance from the origin very nearly

$$y_0 = \frac{\mu}{C},$$

and the axis of x at points whose coördinates are the smallest (without regard to sign) roots of the equations

$$\frac{\mu}{x} + \frac{n'^2 a'^3}{a' - x} = C + \tfrac{3}{2} n'^2 a'^2 - \tfrac{1}{2} n'^2 (a' - x)^2,$$

$$-\frac{\mu}{x} + \frac{n'^2 a'^3}{a' - x} = C + \tfrac{3}{2} n'^2 a'^2 - \tfrac{1}{2} n'^2 (a' - x)^2$$

For the moon these quantities have the values

$$y_0 = 104.408, \qquad x_1 = -109.655, \qquad x_2 = +109.694.$$

This branch then does not differ much from a circle having its centre at the origin, more closely it approximates to the ellipse whose major axis $= x_2 - x_1$, and minor axis $= 2y_0$.

The value of the coördinate z, for the single intersection of the surface with the axis of z above the plane of xy, is given by the single positive root of the equation

$$\frac{\mu}{z} + \frac{n'^2 a'^3}{\sqrt{(a'^2 + z^2)}} = C + n'^2 a'^2.$$

For the moon the numerical value of this root is

$$z_0 = 102.956.$$

The intersection of the surface with the perpendicular to the plane of xy passing through the centre of the sun is, in like manner, given by the equation

$$\frac{\mu}{\sqrt{(a'^2 + z^2)}} + \frac{n'^2 a'^3}{z} = C + \tfrac{1}{2} n'^2 a'^2,$$

having but a single positive root, which is nearly

$$z_0 = \frac{\tfrac{2}{3} a'}{1 + \tfrac{2}{3} \dfrac{C}{n'^2 a'^2}},$$

or, with less exactitude,

$$z_0 = \tfrac{2}{3} a'.$$

From these investigations it is possible to get a tolerably clear idea of the form of this surface. When C exceeds a certain limit, it consists of three separate folds. The first being quite small, relatively to the other two, is closed, surrounds the earth and somewhat resembles an ellipsoid whose axes have been given above. The second is also closed, but surrounds the sun,

and has approximately the form of an ellipsoid of revolution, the semiaxis in the plane of the ecliptic being somewhat less than a', and the semiaxis of revolution perpendicular to the ecliptic and passing through the sun being about two-thirds of this. This fold has a protuberance in the portion neighboring the earth. The third fold is not closed, but is asymptotic to the cylinder mentioned at the beginning of the investigation of the surface. Like the second, it also is nearly of revolution about an axis passing through the centre of the sun and perpendicular to the ecliptic. The radius of its trace on the ecliptic is about as much greater than a', as the radius of the trace of the second fold falls short of that quantity. The fold has a protuberance in the portion neighboring the earth, and which projects towards this body. The whole fold resembles a cylinder bent inwards in a zone neighboring the ecliptic.

What modifications take place in these folds when the constants involved in the equation of the surface are made to vary, will be clearly seen from the following exposition. Let us, for brevity, put

$$h = 3 + 2\frac{C}{n'^2a'^2},$$

and, for the moment, adopt a', the distance of the earth from the sun, as the linear unit, and transfer the origin to the centre of the sun, and moreover put

$$\gamma = \frac{\mu}{n'^2a'^3}.$$

Then the intersections of the surface, with the axis of x, will be given by the two roots of the equation

$$x^4 - x^3 - hx^2 + (h + 2 - 2\gamma)\,x - 2 = 0,$$

which lie between the limits 0 and 1; by the two roots of

$$x^4 - x^3 - hx^2 + (h + 2 + 2\gamma)\,x - 2 = 0,$$

which lie between 1 and $\sqrt{h}$; and by the two roots of

$$x^4 - x^3 - hx^2 + (h - 2 - 2\gamma)\,x + 2 = 0,$$

which lie between 0 and $-\sqrt{h}$.

Hence, if C diminishes so much that the first of these three equations has the two roots, lying between the mentioned limits, equal, the first fold will have a contact with the second fold; and if C fall below this limit, the roots become imaginary, and the two folds become one. Again, if C is diminished to the limit where the second equation has the mentioned pair of roots equal, the first fold will have a contact with the third; and when C is

less than this, these two folds form but one. And when C is less than both these limits, there will be but one fold to the surface.

In the spaces inclosed by the first and second folds the velocity, relative to the moving axes of coördinates, is real; but, in the space lying between these folds and the third fold, it is imaginary; without the third fold it is again real. Thus, in those cases, where C and γ have such values that the three folds exist, if the body, whose motion is considered, is found at any time within the first fold, it must forever remain within it, and its radius vector will have a superior limit. If it be found within the second fold, the same thing is true, but the radius vector will have an inferior as well as a superior limit. And if it be found without the third fold, it must forever remain without, and its radius vector will have an inferior limit.

Applying this theory to our satellite, we see that it is actually within the first fold, and consequently must always remain there, and its distance from the earth can never exceed 109.694 equatorial radii. Thus, the eccentricity of the earth's orbit being neglected, we have a rigorous demonstration of a superior limit to the radius vector of the moon.

In the cases, where C and γ have such values that the surface forms but one fold, Jacobi's integral does not afford any limits to the radius vector.

When we neglect the solar parallax and the lunar inclination, the preceding investigation is reduced to much simpler terms. The surface then degenerates into a plane curve, whose equation, of the sixth degree, is

$$\frac{\mu}{r} = C - \tfrac{3}{2} n'^2 x^2.$$

It is evidently symmetrical with respect to both axes of coördinates, and is contained between the two right lines, whose equations are

$$x = \pm \sqrt{\frac{2C}{3n'^2}},$$

and which are asymptotic to it. It intersects the axis of y, at two points, whose coördinates are

$$y = \pm \frac{\mu}{C}.$$

The cubic equation,

$$r^3 - \frac{2C}{3n'^2} r + \frac{2\mu}{3n'^2} = 0$$

gives the values of r, for which the curves intersect the axis of x. If

$$(2C)^{\frac{3}{2}} > 9\mu n',$$

this equation has two real roots between the limits 0 and $+\sqrt{\frac{2C}{3n'^2}}$. If

$$(2C)^{\frac{3}{2}} = 9\mu n',$$

these roots become equal. And if

$$(2C)^{\frac{3}{2}} < 9\mu n',$$

there are no real roots between these limits, and the curve has no intersection with the axis of x. The figures below exhibit the three varieties of this curve.

Fig. 1

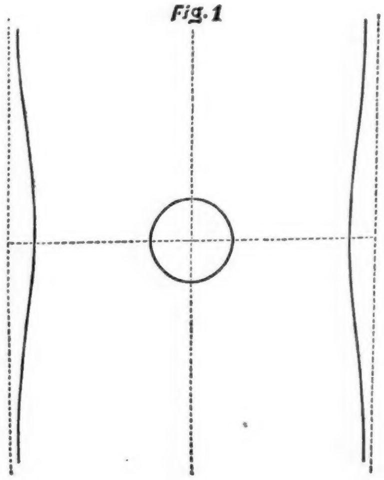

Fig. 2

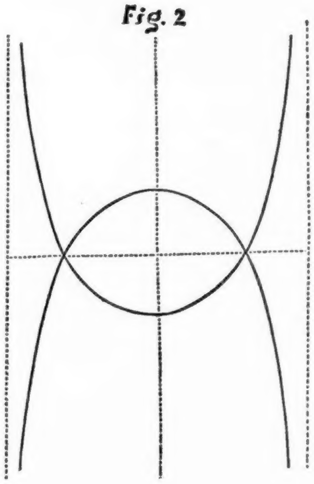

Fig. 3

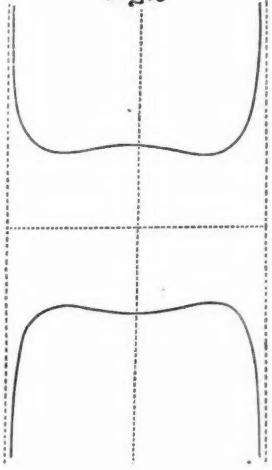

Fig. 1 represents the form of the curve in the case of our moon. In Fig. 2 we see that the small oval of Fig. 1 has enlarged and elongated itself so as to touch the two infinite branches; while, in Fig. 3, it has disappeared, the portions of the curve, lying on either side of the axis of x, having lifted themselves away from it, and the angles having become rounded off. In Fig. 1, the velocity is real within the oval, and also without the infinite branches, but it is imaginary in the portion of the plane lying between the oval and these branches. Hence, if the body be found, at any time, within the oval, it cannot escape thence, and its radius vector will have a superior limit; and, if it be found in one of the spaces on the concave side of the infinite branches, it cannot remove to the other, and its radius vector will have an inferior limit.

In the case represented in Fig. 2, the same things are true, but it seems as if the body might escape from the oval to the infinite spaces, or vice versa, at the points where the curve intersects the axis of x. However, at these points, the force, no less than the velocity, is reduced to zero. For the distance of these points from the origin is the positive root of the equation

$$3r^2 - \frac{2C}{3n'^2} = 0,$$

or

$$\frac{\sqrt{2C}}{3n'} = \frac{\sqrt[3]{9\mu n'}}{3n'},$$

and this value is the same as that given by the equation

$$\frac{\mu}{r^3} - 3n'^2 = 0.$$

In consequence the forces vanish at these two points, and thus we have two particular solutions of our differential equations.*

In the case represented in Fig. 3, the integral does not afford any superior or inferior limit to the radius vector.

The surface, or, in the more simple case, the plane curve, we have discussed, is the locus of zero velocity; and the surface or plane curve, upon which the velocity has a definite value, is precisely of the same character and has a similar equation. It is only necessary to suppose that the C of the preceding formulæ is augmented by half the square of the value attributed to the velocity. Thus, in the case of our moon, it is plain the curves of equal velocity will form a series of ovals surrounding the origin, and approaching it, and becoming more nearly circular as the velocity increases.

Applying the simple formulæ, where the solar parallax is neglected, to the moon, we find that the distance of the asymptotic lines, from the origin, is

$$\sqrt{\frac{2C}{3n'^2}} = 500.4992.$$

The distance of the points on the axis of x, at which the moon would remain stationary with respect to the sun, is

$$\sqrt[3]{\frac{\mu}{3n'^2}} = 235.5971.$$

*The corresponding solution, in the more general problem of three bodies, may be seen in the *Mécanique Céleste, Tom.* IV, p. 310.

If the auxillary angle θ is derived from the equation

$$\sin\theta = \frac{9\mu n'}{(2C)^{\frac{3}{2}}}.$$

we get

$$\theta = 32^\circ\ 49'\ 6''.63\,;$$

and the distances from the origin, at which the curve of zero velocity intersects the axis of x, are given by the two expressions

$$\frac{2\sqrt{2C}}{3n'}\sin\frac{\theta}{3},$$

$$\frac{2\sqrt{2C}}{3n'}\sin\left(60^\circ - \frac{\theta}{3}\right),$$

and the numbers are 109.6772 and 435.5623. These values differ but little from the previous more general determinations.

(*To be continued.*)

THE THEOREM OF THREE MOMENTS.

By Henry T. Eddy, C. E., Ph. D., *University of Cincinnati.*

The theorem expressing the relationship between the moments of flexure of a straight elastic girder at three successive points of support, was first published by Clapeyron in the Comptes Rendus, 1857. The investigation contemplated a girder of uniform cross section, whose points of support were situated upon the same level at any unequal distances apart, and loaded with any uniform loading between the successive points of support, but that loading of different intensity in the different spans into which the points of support divide the girder. The formula has since been generalized by the labors of Winkler, Bresse, Heppel, Weyrauch and others so as to include all cases; viz: when the loading is distributed in any manner whatever, either continuously or discontinuously; when the cross section of the girder changes uniformly or at intervals; when the points of support are at any different heights, and when the girder is fixed at the end supports in such a manner that it can or cannot change its direction at one or both of those points. The formula supposes that the elastic limits are in no case surpassed, either by too great loading or too great difference in height between the points of support, or both.

It is proposed in the present paper to obtain, in a direct manner, a new and simplified equation to express this relationship in its most general form, and to point out at the same time what may be regarded as the proper significance of the quantities appearing in the equation.

Let us assume the ordinary fundamental equations applicable to elastic girders under the action of vertical forces as already proven: they may be stated as follows:

Let O be any point of the girder at which we are to express either the shearing stress, the bending moment, the curvature, the slope, or the deflection.

Let z = the distance from O as an origin to the point of application of any vertical force P.

Let x = the distance from O as an origin to the point of application of any bending moment M.

Let r = the radius of curvature of the deflected girder at O.

Let I = the moment of inertia of the cross section of the girder about its neutral axis at any point.

Let E = the modulus of elasticity of the material, supposed to be constant.

Then the relations we propose to assume are:

$$\left.\begin{array}{l}\text{Force applied at any point is } P.\\ \text{Shearing stress at } O \text{ is } S = \Sigma\,(P).\\ \text{Bending moment at } O \text{ is } M = \Sigma\,(Pz).\end{array}\right\} \quad \ldots\ldots\ldots \quad (1)$$

$$\left.\begin{array}{l}\text{Curvature at } O \text{ is } 1 \div r = R = M \div EI.\\ \text{Slope at } O \text{ is } T = \Sigma\,(R) = \Sigma\,(M \div EI).\\ \text{Deflection at } O \text{ is } D = \Sigma\,(Rx) = \Sigma\,(Mx \div EI).\end{array}\right\} \quad \ldots\ldots \quad (2)$$

It is evident that P bears the same kind of relation to M in (1), that $M \div EI$ does to D in (2). In order to comprehend what the relationship is, it is only necessary to conceive that vertical ordinates be laid off at each point of the span equal to M; the locus of their extremities has been called the "curve of the actual moments," and it is known that it is the same as the catenary of the same depth which supports the weights P. Now from this curve, another curve (or polygon as the case may be) can be described, whose ordinates are equal to $R = M \div EI$, which may be called the "curve of the effective moments," since the amount of bending (*i. e.* difference of slope) and the deflection are dependent upon R. If this curve of effective moments be regarded as the surface of some species of homogenous loading whose depth is R, and a third curve be drawn such that its ordinates are the moments which would be produced by such loading, then the third curve is the "curve of actual deflections," and is the shape which the deflected girder will assume under the action of the loads P. By well-known graphical methods, these curves can all be constructed without the use of algebraic processes, with at least sufficient accuracy for practical computations.

It should be noticed that M is used in two senses: it may signify the ordinate of the moment curve first mentioned; or, it may signify a part of the "moment area" included between that curve and the span and having an horizontal width of one unit. It is in this last sense that it is used in (2), so that if it be desirable to estimate the dimension of the terms in any equation, regard must be had to this fact. It will readily appear in which sense M is used in the expressions hereafter employed, even though both are found in the same equation.

Let I_0 = the moment of inertia about its neutral axis of some particular cross section, which is assumed as the standard of comparison.

Let $i = I_0 \div I$ = the ratio of the standard moment of inertia to that at any other cross section.

Let a, c, a' be three successive piers of a continuous girder of several unequal spans, in the order of their position, c being the intermediate pier.

Let magnitudes in the right hand span be distinguished from the corresponding magnitudes in the left, by primes.

Let l and l' be the length of the spans.

Let t and t' be the trigonometrical tangents of the acute angles at c between the horizontal and the line tangent to the deflection curve.

Let y be the ordinate of any point of the girder above some datum level, *i. e.* above the axis of x.

Then, in case of deflections so small as those occurring in elastic girders, we have sensibly

$$D_a = y_a - y_c - lt \quad , \quad D'_a = y'_a - y_c - l't' \;, \quad . \; . \; . \; . \quad (3)$$

in which the deflections are reckoned from the tangent at c to the deflected girder. Also, the equation of deflections (2) may be written,

$$DEI_0 = \Sigma\,(Mix) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

in which any ordinate M of the moment curve in the span l, may be regarded as consisting, in the aggregate, of three parts: viz, M_1 dependent upon the moment M_a at the pier a, M_2 caused by the weights P in the span itself, and M_3 dependent upon the moment M_c at c. M_1 and M_3 are the effect at any point in the span of the weights P, which are applied to spans other than l. We therefore obtain by (3) for the two spans under consideration, when the summation is extended from c, the intermediate pier, to each of the piers a and a'.

$$\left.\begin{aligned} EI_0\,(y_a - y_c - lt) &= \Sigma_c^a\,[(M_1 + M_2 + M_3)\,ix] \\ EI_0\,(y'_a - y_c - l't') &= \Sigma_c^{a'}\,[(M'_1 + M'_2 + M'_3)\,i'x'] \end{aligned}\right\} \quad . \; . \; . \; . \quad (5)$$

in which x and x' are measured respectively from a and a' towards c.

Now the effect of M_a, which is a bending moment at a due to loads at the left of a, is a moment M_1 which uniformly decreases from a to c, as may be readily seen when we have regard to the fact that M_a is a couple which is held in equilibrium by its increasing (or decreasing) the vertical reaction at c.

The moment due to a vertical reaction at c increases uniformly from c to a, hence

$$M_1 = M_a\,(l - x) \div l, \text{ and } M_3 = M_c x \div l, \quad . \; . \; . \; . \; . \quad (6)$$

Let $$\bar{x} = \Sigma_c^a (Mix) \div \Sigma_c^a (Mi)\,, \quad \ldots \ldots \quad (7)$$
then is $\bar{x}$ the distance from a to the centre of gravity of the effective moment area due to the moments M; $\bar{x}$ can be found with ease by a graphical process. Then, from (6) and (7),

$$\left.\begin{aligned} \bar{x}_1 &= \int_c^a i\,(l-x)\,xdx \div \int_c^a i\,(l-x)\,dx\,, \\ \bar{x}_3 &= \int_c^a i\,x^2dx \div \int_c^a i\,xdx\,. \end{aligned}\right\} \quad \ldots \ldots \quad (8)$$

Again, let $$i_1 = \Sigma_c^a (M_1 i) \div \Sigma_c^a (M_1)\,, \quad \ldots \ldots \quad (9)$$
then is i_1 an average value of i for the moment area due to M_a. Then from (6) and (9),

$$\left.\begin{aligned} i_1 &= \int_c^a i\,(l-x)\,dx \div \int_c^a (l-x)\,dx\,, \\ i_3 &= \int_c^a i\,x\,dx \div \int_c^a xdx. \end{aligned}\right\} \quad \ldots \ldots \quad (10)$$

In similar manner it may be convenient to let $\bar{x}_2$ be the distance of the center of gravity of the effective moment area due to the weights P within the span, and to let i_2 denote the average value of i for the same area; but it is not possible in general to write integrals expressing the values of $\bar{x}_2$ and i_2 in terms of x, until the distribution of the loading is given: they are to be found in any case from equations of the form of (7) and (9).

When the value of i varies discontinuously, it is necessary to divide the limits of the integrals in (8) and (10), so that instead of a single integral we have the sum of several, each extending over a single portion of the span in which i varies continuously.

Once more, $$\left.\begin{aligned} \Sigma_c^a (M_1) &= \tfrac{1}{2} M_a l \\ \Sigma_c^a (M_3) &= \tfrac{1}{2} M_c l, \end{aligned}\right\} \quad \ldots \ldots \quad (11)$$
as appears from previous statements, for these are the expressions for the moment areas due to the moments M_a and M_c respectively.

Equations (5) may now be written

$$\left.\begin{aligned} EI_0\,(y_a - y_c - lt) &= i_2\bar{x}_2\Sigma_c^a (M_2) + \tfrac{1}{2}l\,(M_a i_1 \bar{x}_1 + M_c i_3 \bar{x}_3) \\ EI_0\,(y'_a - y'_c - l't') &= i'_2\bar{x}'_2\Sigma_c^{a'} (M'_2) + \tfrac{1}{2}l'\,(M'_a i'_1 \bar{x}'_1 + M'_c i'_3 \bar{x}'_3) \end{aligned}\right\}. \quad (12)$$

Let $$t + t' = T \quad \ldots \ldots \quad (13)$$
then is T a known constant, for if the girder is straight at c before deflection, then $T = 0$, and in any case T is the acute angle between the spans l and l' when that is small enough to be regarded as sensibly equal to its tangent.

Now divide equations (12) by l and l' respectively and add, then by (13)

$$EI_0\left[\frac{y_a - y_c}{l} + \frac{y'_a - y'_c}{l'} + T\right] = \frac{i_2\bar{x}_2}{l}\Sigma_c^a (M_2) + \frac{i'_2\bar{x}'_2}{l'}\Sigma_c^{a'} (M'_2)$$
$$+ \tfrac{1}{2}\left[M_a i_1 \bar{x}_1 + M_c i_3 \bar{x}_3 + M'_c i'_3 \bar{x}'_3 + M'_a i'_1 \bar{x}'_1\right] \quad \ldots \quad (14)$$

This equation expresses ***the theorem of three moments*** in its most general form: the only unknown quantities in it, for a given girder and given loading, are the moments at the piers $a\ c\ a'$. When there is no constraint at c, then $M_c = M'_c$; and in any case, $M_c - M'_c = C$ the couple introduced by the constraint at c.

The complexity of this formula, as obtained heretofore, has been due to the fact that the first two terms of the second member have been expressed in terms of the weights P, and no adequate method has been proposed for expressing and interpreting the remaining quantities in the second member, with the exception of the moments.

Let us now derive from (14) the equation expressing the theorem in case of an unconstrained girder having a uniform cross section, when the two first terms are stated in terms of the weights P.

In this case $i = 1$, and we have to determine $\bar{x}_2 \Sigma_c^a (M_2)$. Suppose that the area $\Sigma_c^a (M_2)$ is the sum of parts due to several weights P, then the part due to a single weight is

$$M = Pz\,(l - z) \div l\,, \qquad (15)$$

and this may be taken as the height of a triangular moment area whose base is l due to the weight P. This triangle whose area $= \frac{1}{2} M_2 l$ is the part of the moment area due to P, and in computing $\bar{x}_2 \Sigma_c^a (M_2)$ we must find the product of its area by the distance x of its center of gravity from a. Now $x = \frac{1}{3}(l + z)$

$$\therefore \quad \bar{x}_2 \Sigma_c^a (M_2) = \tfrac{1}{6} \Sigma_c^a [P\,(l^2 - z^2)\,z] \qquad (16)$$

Also in this case, $\bar{x}_1 = \frac{1}{3} l\,,\quad \bar{x}_3 = \frac{2}{3} l\,,$

$$\therefore \quad 6EI\left[\frac{y_a - y_c}{l} + \frac{y'_a - y_c}{l'} + T\right] = \frac{1}{l}\Sigma_c^a\Big[P\,(l^2 - z^2)\,z\Big] - \frac{1}{l'}\Sigma_c^{a'}\Big[P\,(l'^2 - z'^2)\,z'\Big]$$
$$+ M_a l + 2\,M_c\,(l + l') + M'_a l'\,, \qquad (17)$$

a well known form.

Again, if the girder is straight at c, the piers on the same level, and the cross section constant, we have

$$6\left[\frac{1}{l} A_2 \bar{x}_2 + \frac{1}{l'} A'_2 \bar{x}'_2\right] = M_a l + 2\,M_c\,(l + l') + M'_a l'\,,$$

a form of the equation first given by Professor Chas. E. Greene, 1875, in which A_2 and A'_2 are the moment areas due to the applied weights.

SOLUTION OF THE IRREDUCIBLE CASE;

Or, to express the three roots of the general and complete equation of the third degree in finite, algebraic and really performable functions of the coëfficients of the equation, when these roots are all real and at least one of them is rational.†*

BY GUIDO WEICHOLD, *of Zittau, Saxony.*

Let a, b, c be the roots of the general and complete equation of the third degree:

$$x^3 + Ax^2 + Bx + C = 0; \text{ and}$$

$$\theta = \frac{-1 + \sqrt{-3}}{2}, \quad \theta' = \frac{-1 - \sqrt{-3}}{2}. \quad \text{Whence,}$$

$$a + \theta b + \theta' c = \frac{(2a - b - c) + (b - c)\sqrt{-3}}{2} = \rho;$$

$$bc + \theta ac + \theta' ab = \frac{(2bc - ac - ab) - a(b - c)\sqrt{-3}}{2} = \rho_1;$$

$$a + \theta' b + \theta c = \frac{(2a - b - c) - (b - c)\sqrt{-3}}{2} = \rho';$$

$$bc + \theta' ac + \theta ab = \frac{(2bc - ac - ab) + a(b - c)\sqrt{-3}}{2} = \rho'_1; \ddagger$$

then from the known relations:

$$a + b + c = -A; \quad ab + ac + bc = B; \quad abc = -C:$$

$$a = \frac{-A + \rho + \rho'}{3} = -\frac{\rho_1 - \rho'_1}{\rho - \rho'} = -\frac{2B - (\rho_1 + \rho'_1)}{2A + (\rho + \rho')} = -\frac{3C}{B + (\rho_1 + \rho'_1)}$$

$$b = \frac{-A + \theta'\rho + \theta\rho'}{3} = -\frac{\theta'\rho_1 - \theta\rho'_1}{\theta'\rho - \theta\rho'} = -\frac{2B - (\theta'\rho_1 + \theta\rho'_1)}{2A + (\theta'\rho + \theta\rho')} = -\frac{3C}{B + (\theta'\rho_1 + \theta\rho'_1)};$$

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* From the above enunciation it is obvious, that the so-called trigonometrical solution of a cubic equation is no more a solution of the irreducible case than the division of an angle into three equal parts by means of a protractor, would be a solution of the problem of the trisection of an angle, as it in reality amounts to no more than the copying of the roots from tables, where their *approximate* values are indirectly given.

† In the case where the three roots are all irrational, it would be as absurd to require these roots to be determined with numerical exactness, as it would be to require a numerically exact root of a number which is an imperfect power of the degree of the root to be extracted. Moreover, the determination of the approximate values of such roots has nothing to do with the problem of the irreducible case, as the operations indicated in Cardan's formula become performable when treated by approximation (expansion into series).

‡ Consequently ρ and ρ', as well as ρ_1 and ρ'_1, are *conjugate* complex functions of a, b, c, *i. e.*, quantities consisting of two parts, one of which is real and the other imaginary.

$$c=\frac{-A+\theta\rho+\theta'\rho'}{3}=-\frac{\theta\rho_1-\theta'\rho'_1}{\theta\rho-\theta'\rho'}=-\frac{2B-(\theta\rho_1+\theta'\rho'_1)}{2A+(\theta\rho+\theta'\rho')},=-\frac{3C}{B+(\theta\rho_1+\theta'\rho'_1)};$$

or written collectively:

$$\left.\begin{matrix}a\\b\\c\end{matrix}\right\}=\frac{-A+\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho+\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'}{3}=-\frac{\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho_1-\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'_1}{\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho-\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'}=-\frac{2B-\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho_1-\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'_1}{2A+\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho+\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'}$$

$$=-\frac{3C}{B+\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho'+\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'_1}.$$

Now the four auxiliary quantities ρ, ρ', ρ_1, ρ'_1 can be expressed in terms of the coëfficients A, B, C of the proposed equation by means of the following seven relations:

1°. $\rho\rho'=\theta\rho.\theta'\rho'=\theta'\rho.\theta\rho'=(a+\theta b+\theta' c)(a+\theta' b+\theta c)=(b+\theta a+\theta' c)(b+\theta' a+\theta c)=(c+\theta a+\theta' b)(c+\theta' a+\theta b)=\frac{1}{4}\{[(a-b)+(a-c)]^2+3(b-c)^2\}=\frac{1}{4}\{[(b-c)-(a-b)]^2+3(a-c)^2\}=\frac{1}{4}\{[(a-c)+(b-c)]^2+3(a-b)^2\}^*=(a-b)^2-(a-b)(a-c)+(a-c)^2=(b-c)^2+(b-c)(a-b)+(a-b)^2=(a-c)^2-(a-c)(b-c)+(b-c)^2=(a^2+b^2+c^2)-(ab+ac+bc)=A^2-3B=N$, for brevity;

2°. $\rho_1\rho'_1=\theta\rho_1.\theta'\rho'_1=\theta'\rho_1.\theta\rho'_1=(bc+\theta ab+\theta' ac)(bc+\theta' ab+\theta ac)=(ac+\theta ab+\theta' bc)(ac+\theta' ab+\theta bc)=(ab+\theta ac+\theta' bc)(ab+\theta' ac+\theta bc)=\frac{1}{4}\{[b(a-c)+c(a-b)]^2+3a^2(b-c)^2\}=\frac{1}{4}\{[c(a-b)-a(b-c)]^2+3b^2(a-c)^2\}=\frac{1}{4}\{[a(b-c)+b(a-c)]^2+3c^2(a-b)^2\}^*=b^2(a-c)^2-bc(a-b)(a-c)+c^2(a-b)^2=c^2(a-b)^2+ac(a-b)(b-c)+a^2(b-c)^2=a^2(b-c)^2-ab(b-c)(a-c)+b^2(a-c)^2=a^2b^2+a^2c^2+b^2c^2-abc(a+b+c)=B^2-3AC=N'$;

3°. $\rho\rho'_1+\rho'\rho_1=\theta\rho.\theta'\rho'_1+\theta'\rho'.\theta\rho_1=\theta'\rho.\theta\rho'_1+\theta\rho'.\theta'\rho_1=(a+\theta b+\theta' c)(bc+\theta' ac+\theta ab)+(a+\theta' b+\theta c)(bc+\theta ac+\theta' ab)=(c+\theta a+\theta' b)(ab+\theta' bc+\theta ac)+(c+\theta' a+\theta b)(ab+\theta bc+\theta' ac)=(b+\theta c+\theta' a)(ac+\theta' ab+\theta bc)+(b+\theta' c+\theta a)(ac+\theta ab+\theta' bc)=6abc+(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2)(\theta+\theta')=9abc-(ab+ac+bc)(a+b+c)=AB-9C=P$;

* These expressions show that N and N' can be reduced to the form: $\alpha^2+3\beta^2=(\alpha+\beta\sqrt{-3})(\alpha-\beta\sqrt{-3})$, when a, b, c are real and integral, and to the form: $\frac{\alpha^2+3\beta^2}{\gamma^2}=\frac{(\alpha+\beta\sqrt{-3})}{\gamma}\frac{(\alpha-\beta\sqrt{-3})}{\gamma}$, when a, b, c are real and fractional, α, β, γ denoting integers.

4°. $\rho^3 + \rho'^3 = (\rho + \rho')(\theta\rho + \theta'\rho')(\theta'\rho + \theta\rho') = (2a - b - c)(2b - a - c)(2c - a - b) = \{(a - b) + (a - c)\}\{(a - c) + (b - c)\}\{(a - b) - (b - c)\} = 2(a^3 + b^3 + c^3) - 3(ab + ac + bc)(a + b + c) + 27\,abc = -2A^3 + 9AB - 27C = 3P - 2AN = M$;

5° $\rho_1^3 + \rho_1'^3 = (\rho_1 + \rho'_1)(\theta\rho_1 + \theta'\rho'_1)(\theta'\rho_1 + \theta\rho'_1) = (2bc - ab - ac)(2ac - ab - bc)(2ab - ac - bc) = \{b(a - c) + c(a - b)\}\{a(b - c) - c(a - b)\}\{a(b - c) + b(a - c)\} = 2(a^3b^3 + a^3c^3 + b^3c^3) - 3abc(ab + ac + bc)(a + b + c) + 27a^2b^2c^2 = 2B^3 - 9ABC + 27C^2 = -3CP + 2BN' = M'$;

6°. $\rho^2\rho_1 + \rho'^2\rho'_1 = (\theta\rho)^2(\theta\rho_1) + (\theta'\rho')^2(\theta'\rho'_1) = (\theta'\rho)^2(\theta'\rho_1) + (\theta\rho')^2(\theta\rho'_1) = (a + \theta b + \theta'c)^2(bc + \theta'ab + \theta ac) + (a + \theta'b + \theta c)^2(bc + \theta ab + \theta'ac) = (c + \theta a + \theta'b)^2(ab + \theta'ac + \theta bc) + (c + \theta'a + \theta b)^2(ab + \theta ac + \theta'bc) = (b + \theta c + \theta'a)^2(ac + \theta'bc + \theta ab) + (b + \theta'c + \theta a)^2(ac + \theta bc + \theta'ab) = -abc(a + b + c) + 4(a^2b^2 + a^2c^2 + b^2c^2) - (a^2 + b^2 + c^2)(ab + ac + bc) = -9AC + 6B^2 - A^2B = 3N' - BN = AP - 2BN$;

7°. $\rho\rho_1^2 + \rho'\rho_1'^2 = (\theta\rho)(\theta\rho_1)^2 + (\theta'\rho')(\theta'\rho'_1)^2 = (\theta'\rho)(\theta'\rho_1)^2 + (\theta\rho')(\theta\rho'_1)^2 = (a + \theta b + \theta'c)(bc + \theta ac + \theta'ab)^2 + (a + \theta'b + \theta c)(bc + \theta'ac + \theta ab)^2 = (c + \theta a + \theta'b)(ab + \theta bc + \theta'ac)^2 + (c + \theta'a + \theta b)(ab + \theta'bc + \theta ac)^2 = (b + \theta c + \theta'a)(ac + \theta'bc + \theta ab)^2 + (b + \theta'c + \theta a)(ac + \theta bc + \theta'ab)^2 = -abc(ab + ac + bc) + 4abc(a^2 + b^2 + c^2) - (a + b + c)(a^2b^2 + a^2c^2 + b^2c^2) = 9BC - 6A^2C + AB^2 = AN' - 3CN = 2AN' - BP$;

From the foregoing seven relations follow:

1°. $\rho\rho'_1 - \rho'\rho_1 = (a + \theta b + \theta'c)(bc + \theta'ac + \theta ab) - (a + \theta'b + \theta c)(bc + \theta ac + \theta'ab) = (a - b)(a - c)(b - c)\sqrt{-3} = \sqrt{(\rho\rho'_1 + \rho'\rho_1)^2 - 4\rho\rho'\rho_1\rho'_1} = \sqrt{P^2 - 4NN'} = S\sqrt{-3}$, putting $\sqrt{\dfrac{4NN' - P^2}{3}} = S$;

2°. $\rho^3 - \rho'^3 = (\rho - \rho')(\theta\rho + \theta'\rho')(\theta'\rho - \theta\rho') = 3(a - b)(a - c)(b - c)\sqrt{-3} = \sqrt{(\rho^3 + \rho'^3)^2 - 4\rho^3\rho'^3} = \sqrt{M^2 - 4N^3} = \sqrt{(3P - 2AN)^2 - 4N^3} = \sqrt{9P^2 - 12APN + 4A^2N^2 - 4N^3} = \sqrt{9P^2 - 4N(N^2 - A^2N + 3AP)} = \sqrt{9P^2 - 4N.9N'} = 3\sqrt{P^2 - 4NN'} = 3S\sqrt{-3}$;

3°. $\rho_1^3 - \rho_1'^3 = (\rho_1 - \rho'_1)(\theta\rho_1 - \theta'\rho'_1)(\theta'\rho_1 - \theta\rho'_1) = -3abc(a - b)(a - c)(b - c)\sqrt{-3} = \sqrt{(\rho_1^3 + \rho_1'^3)^2 - 4\rho_1^3\rho_1'^3} = \sqrt{M'^2 - 4N'^3} = \sqrt{(-3CP + 2BN')^2 - 4N'^3} = \sqrt{9C^2P^2 - 12BCN'P + 4B^2N'^2 - 4N'^3} = \sqrt{9C^2P^2 - 4N'(N'^2 - B^2N' + 3BCP)} = \sqrt{9C^2P^2 - 4N.9C^2N} = 3C\sqrt{P^2 - 4NN'} = 3CS\sqrt{-3}$;

4°. $\rho^2\rho_1 - \rho'^2\rho'_1 = (a + \theta b + \theta' c)^2(bc + \theta' ab + \theta ac) - (a + \theta' b + \theta c)^2(bc + \theta ab + \theta' ac) = (a+b+c)(a-b)(a-c)(b-c)\sqrt{-3} = \sqrt{(\rho^2\rho_1 + \rho'^2\rho'_1)^2 - 4\rho^2\rho'^2\rho_1\rho'_1}$
$= \sqrt{(AP - 2BN)^2 - 4N^2N'} = \sqrt{A^2P^2 - 4ABPN + 4B^2N^2 - 4N^2N'}$
$= \sqrt{A^2P^2 - 4N(NN' - B^2N + ABP)} = \sqrt{A^2P^2 - 4N.A^2N'}$
$= A\sqrt{P^2 - 4NN'} = AS\sqrt{-3}$;

5°. $\rho\rho_1^2 - \rho'\rho_1'^2 = (a + \theta b + \theta' c)(bc + \theta' ab + \theta ac)^2 - (a + \theta' b + \theta c)(bc + \theta ab + \theta' ac)^2$
$= (ab + ac + bc)(a-b)(a-c)(b-c)\sqrt{-3} = \sqrt{(\rho\rho_1^2 + \rho'\rho_1'^2)^2 - 4\rho\rho'\rho_1^2\rho_1'^2}$
$= \sqrt{(2AN' - BP)^2 - 4NN'^2} = \sqrt{4A^2N'^2 - 4ABPN' + B^2P^2 - 4NN'^2}$
$= \sqrt{B^2P^2 - 4N'(NN' - A^2N' + ABP)} = \sqrt{B^2P^2 - 4N'.B^2N}$
$= B\sqrt{P^2 - 4NN'} = BS\sqrt{-3}$;

and, moreover:

$\rho\rho'_1 = \frac{1}{2}\{(\rho\rho'_1 + \rho'\rho_1) + (\rho\rho'_1 - \rho'\rho_1)\} = \frac{1}{2}\{P + \sqrt{P^2 - 4NN'}\} = \frac{1}{2}\{P + S\sqrt{-3}\}$;

$\rho'\rho_1 = \frac{1}{2}\{(\rho\rho'_1 + \rho'\rho_1) - (\rho\rho'_1 - \rho'\rho_1)\} = \frac{1}{2}\{P - \sqrt{P^2 - 4NN'}\} = \frac{1}{2}\{P - S\sqrt{-3}\}$;

$\rho^3 = \frac{1}{2}\{(\rho^3 + \rho'^3) + (\rho^3 - \rho'^3)\} = \frac{1}{2}\{M + \sqrt{M^2 - 4N^3}\} = \frac{1}{2}\{3P - 2AN + 3S\sqrt{-3}\}$;

$\rho'^3 = \frac{1}{2}\{(\rho^3 + \rho'^3) - (\rho^3 - \rho'^3)\} = \frac{1}{2}\{M - \sqrt{M^2 - 4N^3}\} = \frac{1}{2}\{3P - 2AN - 3S\sqrt{-3}\}$;

$\rho_1^3 = \frac{1}{2}\{(\rho_1^3 + \rho_1'^3) + (\rho_1^3 - \rho_1'^3)\} = \frac{1}{2}\{M' + \sqrt{M'^2 - 4N'^3}\} = \frac{1}{2}\{-3CP + 2BN' + 3CS\sqrt{-3}\}$;

$\rho_1'^3 = \frac{1}{2}\{(\rho_1^3 + \rho_1'^3) - (\rho_1^3 - \rho_1'^3)\} = \frac{1}{2}\{M' - \sqrt{M'^2 - 4N'^3}\} = \frac{1}{2}\{-3CP + 2BN' - 3CS\sqrt{-3}\}$;

$\rho^2\rho_1 = \frac{1}{2}\{(\rho^2\rho_1 + \rho'^2\rho'_1) + (\rho^2\rho_1 - \rho'^2\rho'_1)\} = \frac{1}{2}\{3N' - BN + A\sqrt{P^2 - 4NN'}\} = \frac{1}{2}\{AP - 2BN + AS\sqrt{-3}\}$;

$\rho'^2\rho'_1 = \frac{1}{2}\{(\rho^2\rho_1 + \rho'^2\rho'_1) - (\rho^2\rho_1 - \rho'^2\rho'_1)\} = \frac{1}{2}\{3N' - BN - A\sqrt{P^2 - 4NN'}\}$
$= \frac{1}{2}\{AP - 2BN - AS\sqrt{-3}\}$;

$\rho\rho_1^2 = \frac{1}{2}\{(\rho\rho_1^2 + \rho'\rho_1'^2) + (\rho\rho_1^2 - \rho'\rho_1'^2)\} = \frac{1}{2}\{AN' - 3CN + B\sqrt{P^2 - 4NN'}\} = \frac{1}{2}\{2AN' - BP + BS\sqrt{-3}\}$;

$\rho'\rho_1'^2 = \frac{1}{2}\{(\rho\rho_1^2 + \rho'\rho_1'^2) - (\rho\rho_1^2 - \rho'\rho_1'^2)\} = \frac{1}{2}\{AN' - 3CN - B\sqrt{P^2 - 4NN'}\} = \frac{1}{2}\{2AN' - BP - BS\sqrt{-3}\}$.

Now, as these values of $\rho\rho'_1$, $\rho'\rho_1$, ρ^3, ρ'^3, ρ_1^3, $\rho_1'^3$, $\rho^2\rho_1$, $\rho'^2\rho'_1$, $\rho\rho_1^2$, $\rho'\rho_1'^2$, are complex (see note, page 32) when a, b, c are simultaneously real, as appears from the inspection of $S = (a-b)(a-c)(b-c) = \sqrt{\frac{4NN' - P^2}{3}}$, showing that S

is real and rational when a, b, c are real and rational, the determination of ρ, ρ', ρ_1, ρ'_1, by the extraction of the cube root of $\rho^3 = \frac{1}{2}\{M + 3S\sqrt{-3}\}$; $\rho'^3 = \frac{1}{2}\{M - 3S\sqrt{-3}\}$; $\rho_1^3 = \frac{1}{2}\{M' + 3CS\sqrt{-3}\}$; $\rho_1'^3 = \frac{1}{2}\{M' - 3CS\sqrt{-3}\}$, would involve the irreducible case. But the values of the four auxiliary quantities ρ, ρ', ρ_1, ρ'_1, can be found without the operation of extracting the cube root of ρ^3, ρ'^3, ρ_1^3, $\rho_1'^3$, by the determination, in various ways, of the factors respectively common. Thus:

I. to $\rho\rho'$ and $\rho\rho'_1$, viz. ρ; to $\rho_1\rho'_1$ and $\rho'\rho_1$, viz. ρ_1;
" $\rho\rho'$ " $\rho'\rho_1$, " ρ'; " $\rho_1\rho'_1$ " $\rho\rho'_1$, " ρ'_1;

II. to $\rho\rho'$ and ρ^3, viz. ρ; to $\rho_1\rho'_1$ and ρ_1^3, viz. ρ_1;
" $\rho\rho'$ " ρ'^3, " ρ'; " $\rho_1\rho'_1$ " $\rho_1'^3$, " ρ'_1;

III. to $\rho\rho'$ and $\rho^2\rho_1$, viz. ρ; to $\rho_1\rho'_1$ and $\rho^2\rho_1$, viz. ρ_1;
" $\rho\rho'$ " $\rho'^2\rho'_1$, " ρ'; " $\rho_1\rho'_1$ " $\rho'^2\rho'_1$, " ρ'_1;

IV. to $\rho\rho'$ and $\rho\rho_1^2$, viz. ρ; to $\rho_1\rho'_1$ and $\rho\rho_1^2$, viz. ρ_1;
" $\rho\rho'$ " $\rho'\rho_1'^2$, " ρ'; " $\rho_1\rho'_1$ " $\rho'\rho_1'^2$, " ρ'_1;

V. to $\rho\rho'_1$ and $\rho^2\rho_1$, viz. ρ; to $\rho'\rho_1$ and $\rho^2\rho_1$, viz. ρ_1;
" $\rho'\rho_1$ " $\rho'^2\rho'_1$, " ρ'; " $\rho\rho'_1$ " $\rho'^2\rho'_1$, " ρ'_1;

VI. to $\rho\rho'_1$ and $\rho\rho_1^2$, viz. ρ; to $\rho'\rho_1$ and $\rho\rho_1^2$, viz. ρ_1;
" $\rho'\rho_1$ " $\rho'\rho_1'^2$, " ρ'; " $\rho\rho'_1$ " $\rho'\rho_1'^2$, " ρ'_1;

VII. to $\rho\rho'_1$ and ρ^3, viz. ρ; to $\rho'\rho_1$ and ρ_1^3, viz. ρ_1;
" $\rho'\rho_1$ " ρ'^3, " ρ'; " $\rho\rho'_1$ " $\rho_1'^3$, " ρ'_1;

VIII. to $\rho\rho_1^2$ and ρ^3, viz. ρ; to $\rho\rho_1^2$ and ρ_1^3, viz. ρ_1^2, whence $\rho_1 = \sqrt{\rho\rho_1^2 \,\overline{|}\, \rho_1^3}$; *
" $\rho'_1\rho_1'^2$ " ρ'^3, " ρ'; " $\rho'\rho_1'^2$ " $\rho_1'^3$, " $\rho_1'^2$, " $\rho'_1 = \sqrt{\rho'\rho_1'^2 \,\overline{|}\, \rho_1'^3}$;

IX. to $\rho^2\rho_1$ and ρ^3, viz. ρ^2, whence $\rho = \sqrt{\rho^2\rho_1 \,\overline{|}\, \rho^3}$; to $\rho^2\rho_1$ and ρ_1^3, viz. ρ_1;
" $\rho'^2\rho'_1$ " ρ'^3, " ρ'^2, " $\rho' = \sqrt{\rho'^2\rho'_1 \,\overline{|}\, \rho'^3}$; to $\rho'^2\rho'_1$ and $\rho_1'^3$, viz. ρ'_1;

X. to $\rho^2\rho_1$ and $\rho\rho_1^2$, viz. $\rho\rho_1$, whence $\rho = \dfrac{\rho^2\rho_1}{\rho^2\rho_1 \,\overline{|}\, \rho\rho_1^2}$ and $\rho_1 = \dfrac{\rho\rho_1^2}{\rho^2\rho_1 \,\overline{|}\, \rho\rho_1^2}$;
" $\rho'^2\rho'_1$ " $\rho'\rho_1'^2$ " $\rho'\rho'_1$, " $\rho' = \dfrac{\rho'^2\rho'_1}{\rho'^2\rho' \,\overline{|}\, \rho'\rho_1'^2}$ " $\rho'_1 = \dfrac{\rho'\rho_1'^2}{\rho'^2\rho'_1 \,\overline{|}\, \rho'\rho_1'^2}$;

and satisfying at the same time the above seven relations.

Therefore, by means of the sign $\overline{|}$, the roots a, b, c can be symbolically represented as follows:

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* The sign $\overline{|}$ denotes the factor in question common to the quantities on either side of it.

$$\left.\begin{matrix}a\\b\\c\end{matrix}\right\}=\frac{-A+\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho+\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'}{3}=\frac{-A+\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\overline{\rho\rho'|\rho^3}+\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\overline{\rho\rho'|\rho'^3}}{3}$$

$$=-\frac{\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho_1-\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'_1}{\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho-\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'}=-\frac{\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\overline{\rho_1\rho'_1|\rho_1^3}-\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\overline{\rho_1\rho'_1|\rho_1^3}}{\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\overline{\rho\rho'|\rho^3}-\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\overline{\rho\rho'|\rho_1^3}}$$

$$=-\frac{2B-\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho_1-\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'_1}{2A+\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho+\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'}=-\frac{2B-\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\overline{\rho_1\rho'_1|\rho_1^3}-\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\overline{\rho_1\rho'_1|\rho_1'^3}}{2A+\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\overline{\rho\rho'|\rho^3}+\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\overline{\rho\rho'|\rho'^3}}$$

$$=-\frac{3C}{B+\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\rho_1+\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\rho'_1}=-\frac{3C}{B+\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\overline{\rho_1\rho'_1|\rho_1^3}+\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\overline{\rho_1\rho'_1|\rho_1'^3}}$$

and so on, by using for the values of ρ, ρ', ρ_1, ρ'_1 the other combinations specified above for the determination of the common factor in question.

Or, writing the roots in terms of the coëfficients A, B, C:

$$\left.\begin{matrix}a\\b\\c\end{matrix}\right\}=\tfrac{1}{3}\left\{-A+\left.\begin{matrix}1\\\theta'\\\theta\end{matrix}\right\}\overline{A^2-3B\left|\frac{AB-9C+\sqrt{(AB-9C)^2-4(A^2-3B)(B^2-3AC)}}{2}\right.}\right.$$

$$\left.+\left.\begin{matrix}1\\\theta\\\theta'\end{matrix}\right\}\overline{A^2-3B\left|\frac{AB-9C-\sqrt{(AB-9C)^2-4(A^2-3B)(B^2-3AC)}}{2}\right.}\right\}$$

etc.

The determination of the above common factor can be effected, as may be seen from the annexed numerical examples, by means of *a limited number of essentially algebraic operations;* that is to say, ρ, ρ', ρ_1, ρ'_1 are finite, algebraic and really performable functions of the coëfficients A, B, C of the proposed equation, *q. e. f.*

COROLLARY.

The expressions: $a=-\dfrac{\rho_1-\rho'_1}{\rho-\rho'}$; $b=-\dfrac{\theta'\rho_1-\theta\rho'_1}{\theta'\rho-\theta\rho'}=-\dfrac{(\rho_1-\rho'_1)+(\rho_1+\rho'_1)\sqrt{-3}}{(\rho-\rho')+(\rho+\rho')\sqrt{-3}}$;

$$c=-\frac{\theta\rho_1-\theta'\rho'_1}{\theta\rho-\theta'\rho'}=-\frac{(\rho_1-\rho'_1)-(\rho_1+\rho'_1)\sqrt{-3}}{(\rho-\rho')-(\rho+\rho')\sqrt{-3}};$$

$$\rho_1 + \rho'_1 = \frac{P(\rho + \rho') - (\rho - \rho') S\sqrt{-3}}{2N}; \ \rho_1 - \rho'_1 = \frac{P(\rho - \rho') - (\rho + \rho') S\sqrt{-3}}{2N'};$$

(the last two resulting from the combination of: $\rho\rho' = N$; $\rho_1\rho'_1 = N'$; $\rho\rho'_1 = \frac{P + S\sqrt{-3}}{2}$; $\rho'\rho_1 = \frac{P - S\sqrt{-3}}{2}$) furnish, after the elimination of a, $\rho + \rho'$, $\rho - \rho'$, $\rho_1 + \rho'_1$, $\rho_1 - \rho'_1$ on the one hand, and of b, $\rho + \rho'$, $\rho - \rho'$, $\rho_1 + \rho'_1$, $\rho_1 - \rho'_1$ on the other, the two equations:

$$b = -\frac{2N' + (P + S)c}{2Nc + (P - S)}; \quad a = -\frac{2N' + (P - S)c}{2Nc + (P + S)};$$

or, written collectively, the following formula:

$$\left.\begin{matrix} a \\ b \end{matrix}\right\} = -\frac{2N' + (P \pm S)c}{2Nc + (P \mp S)};$$

by means of which, when one root of the proposed equation is known, the remaining two may be found in terms of the known one and the quantities N, N', P and S, which have already been calculated for the determination of the first root.

SCHOLIUM.

The following considerations will show that algebra itself points to the foregoing solution of the irreducible case:

I. The elimination of three of the four quantities ρ, ρ', ρ_1, ρ'_1, of ρ', ρ_1, ρ'_1 for instance, from the four simultaneous equations above obtained:

$$\frac{-A + \rho + \rho'}{3} = -\frac{2B - (\rho_1 + \rho'_1)}{2A + \rho + \rho'}; \ \rho\rho' = N; \ \rho_1\rho'_1 = N'; \ \rho\rho'_1 + \rho'\rho_1 = P;$$

leads to the first equation in ρ:

$$2N\rho^4 - [3P - 2AN - 3S\sqrt{-3}]\,\rho^3 - N[3P - 2AN + 3S\sqrt{-3}]\rho + 2N^3 = 0;$$

which becomes decomposable into a product of two factors by putting in the last term for $N^3 = \rho^3\rho'^3$ its equivalent:

$$\rho^3\rho'^3 = \frac{3P - 2AN + 3S\sqrt{-3}}{2} \cdot \frac{3P - 2AN - 3S\sqrt{-3}}{2}, \text{ viz:}$$

$$\left\{2\rho^3 - (3P - 2AN + 3S\sqrt{-3})\right\}\left\{N\rho - \frac{3P - 2AN - 3S\sqrt{-3}}{2}\right\} = 0;$$

which can be satisfied by putting:

either $2\rho^3-(3P-2AN+3S\sqrt{-3})=0$; or $N\rho-\dfrac{3P-2AN-3S\sqrt{-3}}{2}=0$;

whence $\rho^3=\dfrac{3P-2AN+3S\sqrt{-3}}{2}$; and $\rho=\dfrac{3P-2AN-3S\sqrt{-3}}{2N}=\dfrac{\rho'^3}{\rho\rho'}$.

The value of ρ^3 furnished by the first of these equations coincides with that previously found; the value of ρ furnished by the second, does not satisfy the proposed equation, if the quotient of $\dfrac{3P-2AN-3S\sqrt{-3}}{2}=\rho'^3$ divided by $N=\rho\rho'$ is taken in the ordinary sense, but if taken in the more general sense of *the factor common to the dividend and divisor*, it furnishes the conjugate value to ρ. This interpretation is vindicated by the fact that, as there does not yet exist any sign to denote a factor common to two quantities, that operation cannot manifest itself as the result of other operations, as is the case with addition, subtraction, etc. It would, therefore, be of advantage to add to the notation of the elementary operations, a sign to denote such a common factor, by the adoption of which other important results might likely be obtained.

II. The fact that, though it is impossible to extract the cube root of the values of ρ^3, ρ'^3, ρ_1^3, $\rho_1'^3$ algebraically under finite form, yet the extraction of the cube root of the products $\rho^3\rho'^3$, $\rho_1^3\rho_1'^3$, $\rho^3\rho_1'^3$, $\rho'^3\rho_1^3$, $\rho^6\rho_1^3$, $\rho'^6\rho_1'^3$, $\rho^3\rho_1^6$, $\rho'^3\rho_1'^6$ can be effected in the most general manner, as the values of these cube roots are respectively:

$$\rho\rho'=N;\ \rho_1\rho'_1=N;\ \rho\rho'_1=\frac{P+S\sqrt{-3}}{2};\ \rho'\rho_1=\frac{P-S\sqrt{-3}}{2};$$

$$\rho^2\rho_1=\frac{3N'-BN+AS\sqrt{-3}}{2};\ \rho'^2\rho'_1=\frac{3N'-BN-AS\sqrt{-3}}{2};$$

$$\rho\rho_1^2=\frac{AN'-3CN+BS\sqrt{-3}}{2};\ \rho'\rho_1'^2=\frac{AN'-3CN-BS\sqrt{-3}}{2},$$

is another striking intimation that the determination of ρ, ρ', ρ_1, ρ'_1 is not to be effected in all cases by the extraction of the cube root of the values of ρ^3, ρ'^3, ρ_1^3, $\rho_1'^3$, but in certain cases by the decomposition of $\rho\rho'$, $\rho_1\rho'_1$, $\rho\rho'_1$, $\rho'\rho_1$, $\rho^2\rho_1$, $\rho'^2\rho'_1$, $\rho\rho_1^2$, $\rho'\rho_1'^2$ into their factors, *i. e.*, by the foregoing determination of common factors.

NUMERICAL EXAMPLES

Illustrating the Solution of the Irreducible Case.

I.

$$x^3 - 16x^2 + 73x - 90 = 0.$$
$$A = -16;\ B = 73;\ C = -90.$$

$\rho\rho' = A^2 - 3B = N = 37$; $\rho_1\rho'_1 = N' = 1009$; $\rho\rho'_1 + \rho'\rho_1 = AB - 9C = P = -358$; $\dfrac{\rho\rho'_1 - \rho'\rho_1}{\sqrt{-3}} = (a-b)(a-c)(b-c) = \sqrt{\dfrac{4NN' - P^2}{3}} = S = 84$;
$\rho^3 + \rho'^3 = 3P - 2AN = M = 110$; $\rho_1^3 + \rho_1'^3 = 2BN' - 3CP = M' = 50654$;
$\rho^2\rho_1 + \rho'^2\rho'_1 = 3N' - BN = 326$; $\rho\rho_1^2 + \rho'\rho_1'^2 = AN' - 3CN = -6154$;
$\rho\rho'_1 = \dfrac{P + S\sqrt{-3}}{2} = -179 + 42\sqrt{-3}$; $\rho'\rho_1 = \dfrac{P - S\sqrt{-3}}{2} = -179 - 42\sqrt{-3}$;
$\rho^3 = \dfrac{M + 3S\sqrt{-3}}{2} = 55 + 126\sqrt{-3}$; $\rho'^3 = \dfrac{M - 3S\sqrt{-3}}{2} = 55 - 126\sqrt{-3}$;
$\rho_1^3 = \dfrac{M' + 3CS\sqrt{-3}}{2} = 25327 - 11340\sqrt{-3}$; $\rho_1'^3 = \dfrac{M' - 3CS\sqrt{-3}}{2} = 25327 + 11340\sqrt{-3}$; $\rho^2\rho_1 = \dfrac{3N' - BN + AS\sqrt{-3}}{2} = 163 - 672\sqrt{-3}$;
$\rho'^2\rho'_1 = \dfrac{3N' - BN - AS\sqrt{-3}}{2} = 163 + 672\sqrt{-3}$; $\rho\rho_1^2 = -3077 + 3066\sqrt{-3}$;
$\rho'\rho_1'^2 = -3077 - 3066\sqrt{-3}$.

S being real and rational shows that *a*, *b*, *c* are real and rational, and that the irreducible case occurs here. Therefore ρ, ρ', ρ_1, ρ'_1 cannot be determined otherwise than by the process of "common factors." But as it would be too laborious to exhaust all the possible ways of determining these quantities by that process, the following three combinations are deemed sufficient to illustrate that method.

Determination of the Factor Common:

1. To $\rho\rho' = 37$ and $\rho\rho'_1 = -179 + 42\sqrt{-3}$, viz. ρ:

First operation: $\underbrace{-179 + 42\sqrt{-3}}_{\text{dividend}} : \underbrace{37}_{\text{divisor}} = \underbrace{-5 + \sqrt{-3}}_{\text{quotient}} + \underbrace{\dfrac{6 + 5\sqrt{-3}}{37}}_{\text{remainder}}$

$$= \underbrace{-5 + \sqrt{-3}}_{\text{quotient}} + \sqrt{-3}\,\frac{(5 - 2\sqrt{-3})}{37}$$

$$\text{second operation:}\quad \overset{\text{former divisor}}{37} : \overset{\text{simplified former rem.}}{5 - 2\sqrt{-3}} = \overset{\text{hypothetical quotient}}{\alpha + \beta\sqrt{-3}};\text{*}$$

$$\alpha = 5;\ \beta = 2;\ i.\ e.\ 37 : 5 - 2\sqrt{-3} = 5 + 2\sqrt{-3}.$$

Therefore: $\overline{-179 + 42\sqrt{-3}\,|\,37} = 5 - 2\sqrt{-3}$; hence
$\rho\rho' = (5 - 2\sqrt{-3})\ (5 + 2\sqrt{-3}) = (-5 + 2\sqrt{-3})\ (-5 - 2\sqrt{-3})$ and
$\rho\rho'_1 = (5 - 2\sqrt{-3})\ (-31 - 4\sqrt{-3}) = (-5 + 2\sqrt{-3})(31 + 4\sqrt{-3})$.
To decide whether $\rho = 5 - 2\sqrt{-3}$ or $= -5 + 2\sqrt{-3}$, it suffices to compare the cubes of $5 - 2\sqrt{-3}$ and $-5 + 2\sqrt{-3}$ with the above value of $\rho^3 = 55 + 126\sqrt{-3}$; thus:
$\rho = -5 + 2\sqrt{-3}$; $\rho' = -5 - 2\sqrt{-3}$; $\rho_1 = 31 - 4\sqrt{-3}$; $\rho'_1 = 31 + 4\sqrt{-3}$.

2. To $\rho\rho'_1 = -179 + 42\sqrt{-3}$ and $\rho_1\rho'_1 = 1009$, viz. ρ'_1:

First operation: $\overset{\text{div'd}}{1009} : \overset{\text{div'r}}{-179 + 42\sqrt{-3}} = \overset{\text{hypoth. quot.}}{\alpha + \beta\sqrt{-3}}$; $\alpha = -\dfrac{179}{37}$; $\beta = -\dfrac{42}{37}$; in integers approximately $\alpha = -5$; $\beta = -1$; remainder: $-12 + 31\sqrt{-3} = \sqrt{-3}\,(31 + 4\sqrt{-3})$;

second operation: $\overset{\text{former div'r}}{-179 + 4\sqrt{-3}} : \overset{\text{simplf. former rem.}}{31 + 4\sqrt{-3}} = \overset{\text{hypoth. quot.}}{\alpha + \beta\sqrt{-3}}$; $\alpha = -5$; $\beta = 2$; *i. e.* $-179 + 42\sqrt{-3} : 31 + 4\sqrt{-3} = -5 + 2\sqrt{-3}$. Therefore: $\overline{1009\,|\,-179 + 42\sqrt{-3}} = 31 + 4\sqrt{-3}$; hence $\rho\rho'_1 = (31 + 4\sqrt{-3})\,(-5 + 2\sqrt{-3}) = (-31 - 4\sqrt{-3})\,(5 - 2\sqrt{-3})$;
$\rho_1\rho'_1 = (31 - 4\sqrt{-3})(31 + 4\sqrt{-3}) = (-31 + 4\sqrt{-3})(-31 - 4\sqrt{-3})$.
From the comparison of the cubes of $-5 + 2\sqrt{-3}$ and $5 - 2\sqrt{-3}$ with the value of $\rho^3 = 55 + 126\sqrt{-3}$, we have, as before:
$\rho = -5 + 2\sqrt{-3}$; $\rho' = -5 - 2\sqrt{-3}$; $\rho_1 = 31 - 4\sqrt{-3}$; $\rho'_1 = 31 + 4\sqrt{-3}$.

$$\text{*}\ \frac{m+n\sqrt{-3}}{p+q\sqrt{-3}} = \frac{(m+n\sqrt{-3})(p-q\sqrt{-3})}{(p+q\sqrt{-3})(p-q\sqrt{-3})} = \frac{(mp+3nq)+(np-mq)\sqrt{-3}}{p^2+3q^2} = \frac{mp+3nq}{p^2+3q^2} + \left(\frac{np-mq}{p^2+3q^2}\right)\sqrt{-3}$$

shows that the quotient of the division of one complex quantity by another is in general likewise a complex quantity. If, therefore, this quotient be denoted as above by $\alpha + \beta\sqrt{-3}$, the values of α and β are expressed by the two formulæ:

$$\alpha = \frac{mp+3nq}{p^2+3q^2};\ \beta = \frac{np-mq}{p^2+3q^2},$$

by means of which α and β are calculated in the above as well as in the following examples.

3. To $\rho_1\rho'_1 = 1009$ and $\rho^2\rho_1 = 163 - 672\sqrt{-3}$, viz. ρ_1:
First operation: $1009 : 163 - 672\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = \frac{163}{1369}$; $\beta = \frac{672}{1369}$; or approximately in integers $\alpha = 1$; $\beta = 0$; remainder: $346 + 672\sqrt{-3} = 6\sqrt{-3}\,(112 - 47\sqrt{-3})$;
second operation: $163 - 672\sqrt{-3} : 112 - 47\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = \frac{112}{19}$; $\beta = \frac{-67}{19}$; or nearly $\alpha = 6$; $\beta = -3$; remainder: $-43 - 27\sqrt{-3}$;
third operation: $112 - 47\sqrt{-3} : -43 - 27\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = \frac{1}{4}$; $\beta = \frac{5}{4}$; or nearly: $\alpha = 0$; $\beta = 1$; remainder: $31 - 4\sqrt{-3}$;
fourth operation: $-43 - 27\sqrt{-3} : 31 - 4\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = -1$; $\beta = -1$; *i. e.* $-43 - 27\sqrt{-3} : 31 - 4\sqrt{-3} = -1 - \sqrt{-3}$.
Consequently: $\overline{1009 \mid 163 - 672\sqrt{-3}} = 31 - 4\sqrt{-3}$; whence: $\rho_1\rho'_1 = (31 - 4\sqrt{-3})(31 + 4\sqrt{-3}) = (-31 + 4\sqrt{-3})(-31 - 4\sqrt{-3})$; and $\rho^2\rho_1 = (13 - 20\sqrt{-3})(31 - 4\sqrt{-3}) = (-13 + 20\sqrt{-3})(-31 + 4\sqrt{-3})$. From the comparison of the cubes of $31 - 4\sqrt{-3}$ and $-31 + 4\sqrt{-3}$ with the above value of ρ_1^3, we have: $\rho_1 = 31 - 4\sqrt{-3}$; $\rho'_1 = 31 + 4\sqrt{-3}$; and from the division of the above value of $\rho\rho'_1$ by that of ρ'_1, $\rho = -5 + 2\sqrt{-3}$; $\rho' = -5 - 2\sqrt{-3}$.

Determination of the roots a, b, c:

$$a = \frac{-A + \rho + \rho'}{3} = \frac{16 - 10}{3} = 2 = -\frac{\rho_1 - \rho'_1}{\rho - \rho'} = -\frac{-8\sqrt{-3}}{+4\sqrt{-3}} = 2$$

$$= -\frac{2B - (\rho_1 + \rho'_1)}{2A + (\rho + \rho')} = -\frac{146 - 62}{-32 - 10} = 2 = \frac{-3C}{B + \rho_1 + \rho'_1} = -\frac{-270}{73 + 62} = 2;$$

$$b = \frac{-A + \theta'\rho + \theta\rho'}{3} = \frac{-2A - (\rho + \rho') - (\rho - \rho')\sqrt{-3}}{6} = \frac{32 + 10 - 4\sqrt{-3}\sqrt{-3}}{6} = 9$$

$$= -\frac{\theta'\rho_1 - \theta\rho'_1}{\theta'\rho - \theta\rho'} = -\frac{-(\rho_1 - \rho'_1) - (\rho_1 + \rho'_1)\sqrt{-3}}{-(\rho - \rho') - (\rho + \rho')\sqrt{-3}} = -\frac{8\sqrt{-3} - 62\sqrt{-3}}{-4\sqrt{-3} + 10\sqrt{-3}} = 9$$

$$= -\frac{2B - (\theta'\rho_1 + \theta\rho'_1)}{2A + (\theta'\rho + \theta\rho')} = -\frac{4B + (\rho_1 + \rho'_1) + (\rho_1 - \rho'_1)\sqrt{-3}}{4A - (\rho + \rho') - (\rho - \rho')\sqrt{-3}} = -\frac{292 + 62 + 24}{-64 + 10 + 12} = 9$$

$$= -\frac{3C}{B + \theta'\rho_1 + \theta\rho'_1} = -\frac{6C}{2B - (\rho_1 + \rho'_1) - (\rho_1 - \rho'_1)\sqrt{-3}} = -\frac{-540}{146 - 62 - 24} = 9;$$

$$c=\frac{-A+\theta\rho+\theta'\rho'}{3}=\frac{-2A-(\rho+\rho')+(\rho-\rho')\sqrt{-3}}{6}=\frac{32+10+4\sqrt{-3}\sqrt{-3}}{6}=5$$

$$=-\frac{\theta\rho_1-\theta'\rho'_1}{\theta\rho-\theta'\rho'}=-\frac{-(\rho_1-\rho'_1)+(\rho_1+\rho'_1)\sqrt{-3}}{-(\rho-\rho')+(\rho+\rho')\sqrt{-3}}=-\frac{8\sqrt{-3}+62\sqrt{-3}}{-4\sqrt{-3}-10\sqrt{-3}}=5$$

$$=-\frac{2B-(\theta\rho_1+\theta'\rho'_1)}{2A+(\theta\rho+\theta'\rho')}=-\frac{4B+(\rho_1+\rho'_1)-(\rho_1-\rho'_1)\sqrt{-3}}{4A-(\rho+\rho')+(\rho-\rho')\sqrt{-3}}=-\frac{292+62-24}{-64+10-12}=5$$

$$=-\frac{3C}{B+\theta\rho_1+\theta'\rho'_1}=-\frac{6C}{2B-(\rho_1+\rho'_1)+(\rho_1-\rho'_1)\sqrt{-3}}=-\frac{-540}{146-62+24}=5.$$

Moreover, any two of the roots can be found from the third by means of the formula given in the text: $\left.\begin{matrix}a\\b\end{matrix}\right\}=-\frac{2N'+(P\mp S)\,c}{2Nc+(P\pm S)}$; thus:

1° from $c=5$; $a=-\frac{2018-442\cdot5}{74\cdot5-274}=\frac{192}{96}=2$; and

$$b=-\frac{2018-274\cdot5}{74\cdot5-442}=-\frac{648}{-72}=9;$$

2° from $b=9$; $a=-\frac{2018-274\cdot9}{74\cdot9-442}=-\frac{-448}{224}=2$; and

$$c=-\frac{2018-442\cdot9}{74\cdot9-274}=-\frac{-1960}{392}=5;$$

3° from $a=2$; $b=-\frac{2018-442\cdot2}{74\cdot2-274}=-\frac{1134}{-126}=9$; and

$$c=-\frac{2018-274\cdot2}{74\cdot2-442}=-\frac{1470}{-294}=5.$$

II.

$$x^3+\frac{41}{140}x^2-\frac{79}{14}x+\frac{429}{140}=0;$$

$$A=\frac{41}{140};\ B=-\frac{79}{14};\ C=\frac{429}{140};$$

$\rho\rho'=A^2-3B=N=\frac{333481}{19600}$; $\rho_1\rho'_1=B^2-3AC=N'=\frac{571333}{19600}$; $\rho\rho'_1+\rho'\rho_1$ $=AB-9C=P=-\frac{572930}{19600}$; $\frac{\rho\rho'_1-\rho'\rho_1}{\sqrt{-3}}=\sqrt{\frac{4NN'-P^2}{3}}=S=\frac{380292}{19600}$;

$$\rho\rho'_1 = \frac{P + S\sqrt{-3}}{2} = \frac{-286465 + 190146\sqrt{-3}}{19600};\ \rho\rho_1 = \frac{P - S\sqrt{-3}}{2}$$

$$= \frac{-286465 - 190146\sqrt{-3}}{19600};$$

$$\rho^3 = \frac{3P - 2AN + 3S\sqrt{-3}}{2} = \frac{-13398021 + 79861320\sqrt{-3}}{2744000};$$

$$\rho'^3 = \frac{3P - 2AN - 3S\sqrt{-3}}{2} = \frac{-13398021 - 79861320\sqrt{-3}}{2744000};$$

$$\rho_1^3 = \frac{-82672615 + 244717902\sqrt{-3}}{2744000};\ \rho_1'^3 = \frac{-82672615 - 244717902\sqrt{-3}}{2744000}.$$

N and N' being fractional and S moreover real and fractional, show that the three roots of this equation are real, rational and fractional; therefore the irreducible case occurs again in this example. From the fact that, when a, b, c are real, ρ, ρ' and ρ_1, ρ'_1 are conjugate complex numbers, and that, when moreover a, b, c are rational but fractional ρ, ρ' and ρ_1, ρ'_1 are of the form $\frac{\alpha + \beta\sqrt{-3}}{\gamma}$ and $\frac{\alpha - \beta\sqrt{-3}}{\gamma}$, it follows that $N = \rho\rho'$ and $N' = \rho_1\rho'_1$ are capable of being brought to the form $\frac{\alpha^2 + 3\beta^2}{\gamma^2}$, and that consequently the common denominators of ρ and ρ', as well as of ρ_1 and ρ'_1, are respectively equal to the square-roots of the denominators of N and N', when reduced to their lowest terms. Whence the denominator of the values of $\rho\rho'$ and $\rho_1\rho'_1$ must be equal to the product of the square-roots of the denominators of N and N'. Accordingly these denominators may be set aside in the determination of the quantities ρ, ρ', ρ_1, ρ'_1 by the process of "common factors," and supplied afterwards to the factors common to the numerators of those quantities.

Determination of the factor common:

to $\rho\rho' = \frac{333481}{19600}$ and $\rho_1\rho'_1 = \frac{-286465 + 190146\sqrt{-3}}{19600}$, viz. ρ.

First operation: $-286465 + 190146\sqrt{-3} : 333481$

$$= -1 + \frac{47016 + 190146\sqrt{-3}}{333481} = -1 + 6\sqrt{-3}\,\frac{(31691 - 2612\sqrt{-3})}{333481};$$

second operation: $333481 : 31691 - 2612\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = \frac{31691}{3073}$ $\beta = \frac{2612}{3073}$; in integers nearly $\alpha = 10$; $\beta = 1$; remainder: $8735 - 557\sqrt{-3}$;

third operation: $31691 - 2612\sqrt{-3} : 8735 - 557\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = \frac{961}{508}$; $\beta = \frac{461}{508}$; nearly $\alpha = 2$; $\beta = 1$; remainder: $-2492 - 205\sqrt{-3}$;

fourth operation: $8735 - 557\sqrt{-3} : -2492 - 205\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = -\frac{55}{19}$; $\beta = \frac{47}{19}$; nearly $\alpha = -3$; $\beta = 2$; remainder: $29 - 1202\sqrt{-3}$;

fifth operation: $-2492 - 205\sqrt{-3} : 29 - 1202\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = \frac{2}{13}$; $\beta = -\frac{9}{13}$; nearly $\alpha = 0$; $\beta = -1$; remainder: $1114 - 176\sqrt{-3}$ $= 2(557 - 88\sqrt{-3})$;

sixth operation: $29 - 1202\sqrt{-3}$; $557 - 88\sqrt{-3} = \alpha + \beta\sqrt{-3}$; $\alpha = 1$; $\beta = -2$; *i. e.* $29 - 1202\sqrt{-3}$; $557 - 88\sqrt{-3} = 1 - 2\sqrt{-3}$;

therefore: $\overline{\frac{-286465 + 190146\sqrt{-3}}{19600} \Big| \frac{333481}{19600}} = \frac{557 - 88\sqrt{-3}}{140}$. Whence

$$\rho\rho' = \frac{(557 - 88\sqrt{-3})}{140}\frac{(557 + 88\sqrt{-3})}{140} = \frac{(-557 + 88\sqrt{-3})}{140}\frac{(-557 - 88\sqrt{-3})}{140};$$

$$\rho\rho'_1 = \frac{(557 - 88\sqrt{-3})}{140}\frac{(-629 + 242\sqrt{-3})}{140} = \frac{(-557 + 88\sqrt{-3})}{140}\frac{(629 - 242\sqrt{-3})}{140};$$

and after a verification similar to that in the preceding example, by comparing the cubes of $\frac{557 - 88\sqrt{-3}}{140}$ and $\frac{-557 + 88\sqrt{-3}}{140}$ with the value of ρ^3 $= \frac{3P - 2AN + 3S\sqrt{-3}}{2} = \frac{-133988021 + 79861320\sqrt{-3}}{2744000}$, we find

$$\rho = \frac{-557 + 88\sqrt{-3}}{140};\ \rho' = \frac{-557 - 88\sqrt{-3}}{140};\ \rho_1 = \frac{629 + 242\sqrt{-3}}{140};$$

$$\rho'_1 = \frac{629 - 242\sqrt{-3}}{140}.$$

Determination of the roots a, b, c:

$$a = \frac{-A + \rho + \rho'}{3} = \frac{-41 - 2\cdot 557}{3\cdot 140} = -\frac{11}{4} = -\frac{\rho_1 - \rho'_1}{\rho - \rho'} = -\frac{2\cdot 242\sqrt{-3}}{2\cdot 88\sqrt{-3}}$$

$$= -\frac{11}{4} = -\frac{2B-(\rho_1+\rho'_1)}{2A+(\rho+\rho')} = -\frac{-1580-2\cdot629}{82-2\cdot557} = -\frac{11}{4} = -\frac{3C}{B+\rho_1+\rho_1}$$

$$= -\frac{1287}{-790+2\cdot629} = -\frac{11}{4};$$

$$b = \frac{-A+\theta'\rho+\theta\rho'}{3} = \frac{-2A-(\rho+\rho')-(\rho-\rho')\sqrt{-3}}{6} = \frac{-82+2\cdot557+6\cdot88}{6\cdot140}$$

$$= \frac{13}{7} = -\frac{\theta'\rho_1-\theta\rho'_1}{\theta'\rho-\theta\rho'} = -\frac{-(\rho_1-\rho'_1)-(\rho_1+\rho'_1)\sqrt{-3}}{-(\rho-\rho')-(\rho+\rho')\sqrt{-3}} = -\frac{-2\cdot242\sqrt{-3}-2\cdot629\sqrt{-3}}{-2\cdot88\sqrt{-3}+2\cdot557\sqrt{-3}}$$

$$= \frac{13}{7} = -\frac{2B-(\theta'\rho_1+\theta\rho'_1)}{2A+(\theta'\rho+\theta\rho')} = -\frac{4B+(\rho_1+\rho'_1)+(\rho_1-\rho'_1)\sqrt{-3}}{4A-(\rho+\rho')-(\rho-\rho')\sqrt{-3}}$$

$$= -\frac{-3160+2\cdot629-6\cdot242}{164+2\cdot557+6\cdot88} = \frac{13}{7} = -\frac{3C}{B+\theta'\rho_1+\theta\rho'_1}$$

$$= -\frac{6C}{2B-(\rho_1+\rho'_1)-(\rho_1-\rho'_1)\sqrt{-3}} = -\frac{6\cdot429}{-1580-2\cdot629+6\cdot242} = \frac{13}{7};$$

$$c = \frac{-A+\theta\rho+\theta'\rho'}{3} = \frac{-2A-(\rho+\rho')+(\rho-\rho')\sqrt{-3}}{6} = \frac{-82+2\cdot557-6\cdot88}{6\cdot140} = \frac{3}{5}$$

$$= -\frac{\theta\rho_1-\theta'\rho'_1}{\theta\rho-\theta'\rho'} = -\frac{-(\rho_1-\rho'_1)+(\rho_1+\rho'_1)\sqrt{-3}}{-(\rho-\rho')+(\rho+\rho')\sqrt{-3}} = -\frac{-2\cdot242\sqrt{-3}+2\cdot629\sqrt{-3}}{-2\cdot88\sqrt{-3}-2\cdot557\sqrt{-3}}$$

$$= \frac{3}{5} = -\frac{2B-(\theta\rho_1+\theta'\rho'_1)}{2A+(\theta\rho+\theta'\rho')} = -\frac{4B+(\rho_1+\rho'_1)-(\rho_1-\rho'_1)\sqrt{-3}}{4A-(\rho+\rho')+(\rho-\rho')\sqrt{-3}}$$

$$= -\frac{-3160+2\cdot629+6\cdot242}{164+2\cdot557-6\cdot88} = \frac{3}{5} = -\frac{3C}{B+\theta\rho_1+\theta'\rho'_1} = -\frac{6C}{2B-(\rho_1+\rho'_1)+(\rho_1-\rho'_1)\sqrt{-3}}$$

$$= -\frac{6\cdot429}{-1580-2\cdot629-6\cdot242} = \frac{3}{5}.$$

III.

$$x^3+\frac{81}{55}x^2+\frac{181}{385}x+\frac{3}{77}=0;$$

$$A=\frac{81}{55};\quad B=\frac{181}{385};\quad C=\frac{3}{77};$$

$$N=\frac{16062}{21175}=\frac{112434}{148225};\quad N'=\frac{7246}{148225};\quad P=\frac{50652}{148225};\quad S=\frac{932\sqrt{266}}{148225};$$

$$\rho\rho'_1=\frac{25326+466\sqrt{266}\sqrt{-3}}{148225};\quad \rho'\rho_1=\frac{25326-466\sqrt{266}\sqrt{-3}}{148225};$$

$$\rho^3=\frac{-34498548+538230\sqrt{266}\sqrt{-3}}{57066625};\quad \rho'^3=\frac{-34498548-538230\sqrt{266}\sqrt{-3}}{57066625};$$

$$\rho_1^3=\frac{171856+20970\sqrt{266}\sqrt{-3}}{57066625};\quad \rho'^3_1=\frac{171856-20970\sqrt{266}\sqrt{-3}}{57066625}.$$

S being real but irrational shows that a, b, c are all real, and either all irrational or one rational and the other two irrational, as appears from the formula: $\left.\begin{matrix}a\\ b\end{matrix}\right\} = -\dfrac{2N' + (P \pm S)c}{2Nc + (P \mp S)}$, where N, N', P are rational and S irrational in the present case.

The composition of N, N', P in terms of a, b, c* shows, moreover, that in the latter case N and N' are capable of being brought to the form: $\dfrac{\alpha^2 + 3\lambda\beta^2}{\gamma^2} = \dfrac{\alpha + \beta\sqrt{\lambda}\sqrt{-3}}{\gamma} \cdot \dfrac{\alpha - \beta\sqrt{\lambda}\sqrt{-3}}{\gamma}$; ($\lambda$ being the radical of the irrational factor in S) and indeed:

$$N = \rho\rho' = \frac{126^2 + 3\cdot266\cdot11^2}{385^2} = \frac{(126 \pm 11\sqrt{266}\sqrt{-3})}{385}\,\frac{(126 \mp 11\sqrt{266}\sqrt{-3})}{385}$$

$$= \frac{(-126 \mp 11\sqrt{266}\sqrt{-3})}{385}\,\frac{(-126 \pm 11\sqrt{266}\sqrt{-3}}{385};\quad N' = \rho_1\rho'_1 = \frac{8^2 + 3\cdot266\cdot3^2}{385^2}$$

$$= \frac{(8 \pm 3\sqrt{266}\sqrt{-3})}{385}\,\frac{(8 \mp 3\sqrt{266}\sqrt{-3})}{385} = \frac{(-8 \mp 3\sqrt{266}\sqrt{-3})}{385}$$

$$\frac{(-8 \pm 3\sqrt{266}\sqrt{-3})}{385};\ \text{whence}\ \rho = \frac{126 \pm 11\sqrt{266}\sqrt{-3}}{385}\ \text{or} = \frac{-126 \mp 11\sqrt{266}\sqrt{-3}}{385};$$

$$\rho' = \frac{126 \mp 11\sqrt{266}\sqrt{-3}}{385}\ \text{or} = \frac{-126 \pm 11\sqrt{266}\sqrt{-3}}{385};\ \rho_1 = \frac{8 \pm 3\sqrt{266}\sqrt{-3}}{385}$$

$$\text{or} = \frac{-8 \mp 3\sqrt{266}\sqrt{-3}}{385};\ \rho'_1 = \frac{8 \mp 3\sqrt{266}\sqrt{-3}}{385}\ \text{or} = \frac{-8 \pm 3\sqrt{266}\sqrt{-3}}{385}.$$

Now as the cubes of $\dfrac{126 - 11\sqrt{266}\sqrt{-3}}{385}$, $\dfrac{126 + 11\sqrt{266}\sqrt{-3}}{385}$, $\dfrac{-8 - 3\sqrt{266}\sqrt{-3}}{385}$, $\dfrac{-8 + 3\sqrt{266}\sqrt{-3}}{385}$ are identical with the above values of ρ^3, ρ'^3, ρ_1^3, $\rho_1'^3$, it is evident that these numbers are the values of ρ, ρ', ρ_1, ρ'_1, respectively.

Determination of the roots a, b, c:

$$a = \frac{-A + \rho + \rho'}{3} = \frac{-7\cdot81 + 2\cdot126}{3\cdot385} = -\frac{3}{11} = -\frac{\rho_1 - \rho'_1}{\rho - \rho'} = -\frac{-2\cdot3\sqrt{266}\sqrt{-3}}{-2\cdot11\sqrt{266}\sqrt{-3}}$$

$$= -\frac{3}{11} = -\frac{2B - (\rho_1 + \rho'_1)}{2A + (\rho + \rho')} = -\frac{2\cdot181 + 2\cdot8}{2\cdot7\cdot81 + 2\cdot126} = -\frac{3}{11} = -\frac{3C}{B + \rho_1 + \rho'_1}$$

$$= -\frac{5\cdot3\cdot3}{181 - 2\cdot8} = -\frac{3}{11};$$

* See note, page 33.

$$b=\frac{-A+\theta'\rho+\theta\rho'}{3}=\frac{-2A-(\rho+\rho')-(\rho-\rho')\sqrt{-3}}{6}$$

$$=\frac{-2\cdot7\cdot81-2\cdot126+2\cdot11\sqrt{266}\sqrt{-3}\sqrt{-3}}{6\cdot385}=-\frac{21+\sqrt{266}}{35}=-\frac{3\sqrt{7}+\sqrt{38}}{5\sqrt{7}}$$

$$=-\frac{\theta'\rho_1-\theta\rho'_1}{\theta'\rho-\theta\rho'}=-\frac{-(\rho_1-\rho'_1)-(\rho_1+\rho'_1)\sqrt{-3}}{-(\rho-\rho')-(\rho+\rho')\sqrt{-3}}=-\frac{2\cdot3\sqrt{266}\sqrt{-3}+2\cdot8\sqrt{-3}}{2\cdot11\sqrt{266}\sqrt{-3}-2\cdot12\sqrt{-3}}$$

$$=-\frac{21+\sqrt{266}}{35}=-\frac{3\sqrt{7}+\sqrt{38}}{5\sqrt{7}}=-\frac{2B-(\theta'\rho_1+\theta\rho'_1)}{2A+(\theta'\rho+\theta\rho')}$$

$$=-\frac{4B+(\rho_1+\rho'_1)+(\rho_1-\rho'_1)\sqrt{-3}}{4A-(\rho+\rho')-(\rho-\rho')\sqrt{-3}}=-\frac{4\cdot181-2\cdot8-2\cdot3\sqrt{266}\sqrt{-3}\sqrt{-3}}{4\cdot7\cdot81-2\cdot126+2\cdot11\sqrt{266}\sqrt{-3}\sqrt{-3}}$$

$$=-\frac{118+3\sqrt{266}}{336-11\sqrt{266}}=-\frac{21+\sqrt{266}}{35}=-\frac{3\sqrt{7}+\sqrt{38}}{5\sqrt{7}}=-\frac{3C}{B+\theta'\rho_1+\theta\rho'_1}$$

$$=-\frac{6C}{2B-(\rho_1+\rho'_1)-(\rho_1-\rho'_1)\sqrt{-3}}=-\frac{5\cdot18}{2\cdot181+2\cdot8+2\cdot3\sqrt{266}\sqrt{-3}\sqrt{-3}}$$

$$=-\frac{5}{21-\sqrt{266}}=-\frac{21+\sqrt{266}}{35}=-\frac{3\sqrt{7}+\sqrt{38}}{5\sqrt{7}};$$

$$c=\frac{-A+\theta\rho+\theta'\rho'}{3}=\frac{-2A-(\rho+\rho')+(\rho-\rho')\sqrt{-3}}{6}$$

$$=\frac{-2\cdot7\cdot81-2\cdot126-2\cdot11\sqrt{266}\sqrt{-3}}{6\cdot385}=-\frac{21-\sqrt{266}}{35}=-\frac{3\sqrt{7}-\sqrt{38}}{5\sqrt{7}}$$

$$=-\frac{\theta\rho_1-\theta'\rho'_1}{\theta\rho-\theta'\rho'}=-\frac{-(\rho_1-\rho'_1)+(\rho_1+\rho'_1)\sqrt{-3}}{-(\rho-\rho')+(\rho+\rho')\sqrt{-3}}=-\frac{2\cdot3\sqrt{266}\sqrt{-3}-2\cdot8\sqrt{-3}}{2\cdot11\sqrt{266}\sqrt{-3}+2\cdot126\sqrt{-3}}$$

$$=-\frac{21-\sqrt{266}}{35}=-\frac{3\sqrt{7}-\sqrt{38}}{5\sqrt{7}}=-\frac{2B-(\theta\rho_1+\theta'\rho'_1)}{2A+(\theta\rho+\theta'\rho')}$$

$$=-\frac{4B+(\rho_1+\rho'_1)-(\rho_1-\rho'_1)\sqrt{-3}}{4A-(\rho+\rho')+(\rho-\rho')\sqrt{-3}}=-\frac{4\cdot181-2\cdot8+2\cdot3\sqrt{266}\sqrt{-3}\sqrt{-3}}{4\cdot7\cdot81-2\cdot126-2\cdot11\sqrt{266}\sqrt{-3}\sqrt{-3}}$$

$$=-\frac{118-3\sqrt{266}}{336+11\sqrt{266}}=-\frac{21-\sqrt{266}}{35}=-\frac{3\sqrt{7}-\sqrt{38}}{5\sqrt{7}}=-\frac{3C}{B+\theta\rho_1+\theta'\rho'_1}$$

$$=-\frac{6C}{2B-(\rho_1+\rho'_1)+(\rho_1-\rho'_1)\sqrt{-3}}=-\frac{5\cdot18}{2\cdot181+2\cdot8-2\cdot3\sqrt{266}\sqrt{-3}\sqrt{-3}}$$

$$=-\frac{5}{21+\sqrt{266}}=-\frac{21-\sqrt{266}}{35}=-\frac{3\sqrt{7}-\sqrt{38}}{5\sqrt{7}}.$$

APPENDIX.

The following table of the values of $\alpha^2 + 3\beta^2$, corresponding to all integral values of α and β from 0 to 9, will be found convenient for the determination of $\rho, \rho', \rho_1, \rho'_1$ by the decomposition of $N = \rho\rho'$ and $N' = \rho_1\rho'_1$ into their factors ρ, ρ' and ρ_1, ρ'_1 in case N and N' are tractable numbers.

	$\alpha=0$	$\alpha=1$	$\alpha=2$	$\alpha=3$	$\alpha=4$	$\alpha=5$	$\alpha=6$	$\alpha=7$	$\alpha=8$	$\alpha=9$
$\beta=0$	$0^2+3.0^2=0$	$1^2+3.0^2=1$	$2^2+3.0^2=4$	$3^2+3.0^2=9$	$4^2+3.0^2=16$	$5^2+3.0^2=25$	$6^2+3.0^2=36$	$7^2+3.0^2=49$	$8^2+3.0^2=64$	$9^2+3.0^2=81$
$\beta=1$	$0^2+3.1^2=3$	$1^2+3.1^2=4$	$2^2+3.1^2=7$	$3^2+3.1^2=12$	$4^2+3.1^2=19$	$5^2+3.1^2=28$	$6^2+3.1^2=39$	$7^2+3.1^2=52$	$8^2+3.1^2=67$	$9^2+3.1^2=84$
$\beta=2$	$0^2+3.2^2=12$	$1^2+3.2^2=13$	$2^2+3.2^2=16$	$3^2+3.2^2=21$	$4^2+3.2^2=28$	$5^2+3.2^2=37$	$6^2+3.2^2=48$	$7^2+3.2^2=61$	$8^2+3.2^2=76$	$9^2+3.2^2=93$
$\beta=3$	$0^2+3.3^2=27$	$1^2+3.3^2=28$	$2^2+3.3^2=31$	$3^2+3.3^2=36$	$4^2+3.3^2=43$	$5^2+3.3^2=52$	$6^2+3.3^2=63$	$7^2+3.3^2=76$	$8^2+3.3^2=91$	$9^2+3.3^2=108$
$\beta=4$	$0^2+3.4^2=48$	$1^2+3.4^2=49$	$2^2+3.4^2=52$	$3^2+3.4^2=57$	$4^2+3.4^2=64$	$5^2+3.4^2=73$	$6^2+3.4^2=84$	$7^2+3.4^2=97$	$8^2+3.4^2=112$	$9^2+3.4^2=129$
$\beta=5$	$0^2+3.5^2=75$	$1^2+3.5^2=76$	$2^2+3.5^2=79$	$3^2+3.5^2=84$	$4^2+3.5^2=91$	$5^2+3.5^2=100$	$6^2+3.5^2=111$	$7^2+3.5^2=124$	$8^2+3.5^2=139$	$9^2+3.5^2=156$
$\beta=6$	$0^2+3.6^2=108$	$1^2+3.6^2=109$	$2^2+3.6^2=112$	$3^2+3.6^2=117$	$4^2+3.6^2=124$	$5^2+3.6^2=133$	$6^2+3.6^2=144$	$7^2+3.6^2=157$	$8^2+3.6^2=172$	$9^2+3.6^2=189$
$\beta=7$	$0^2+3.7^2=147$	$1^2+3.7^2=148$	$2^2+3.7^2=151$	$3^2+3.7^2=156$	$4^2+3.7^2=163$	$5^2+3.7^2=172$	$6^2+3.7^2=183$	$7^2+3.7^2=196$	$8^2+3.7^2=211$	$9^2+3.7^2=228$
$\beta=8$	$0^2+3.8^2=192$	$1^2+3.8^2=193$	$2^2+3.8^2=196$	$3^2+3.8^2=201$	$4^2+3.8^2=208$	$5^2+3.8^2=217$	$6^2+3.8^2=228$	$7^2+3.8^2=241$	$8^2+3.8^2=256$	$9^2+3.8^2=273$
$\beta=9$	$0^2+3.9^2=243$	$1^2+3.9^2=244$	$2^2+3.9^2=247$	$3^2+3.9^2=252$	$4^2+3.9^2=259$	$5^2+3.9^2=268$	$6^2+3.9^2=279$	$7^2+3.9^2=292$	$8^2+3.9^2=307$	$9^2+3.9^2=324$

DESIDERATA AND SUGGESTIONS.

By Professor Cayley, *Cambridge, England.*

No. 1.—THE THEORY OF GROUPS.

Substitutions, and (in connexion therewith) groups, have been a good deal studied; but only a little has been done towards the solution of the general problem of groups. I give the theory so far as is necessary for the purpose of pointing out what appears to me to be wanting.

Let α, β, . . . be functional symbols, each operating upon one and the same number of letters and producing as its result the same number of functions of these letters; for instance, $\alpha(x, y, z) = (X, Y, Z)$, where the capitals denote each of them a given function of (x, y, z).

Such symbols are susceptible of repetition and of combination; $\alpha^2(x, y, z) = \alpha(X, Y, Z)$, or $\beta\alpha(x, y, z) = \beta(X, Y, Z)$, $=$ in each case three given functions of (x, y, z), and similarly α^3, $\alpha^2\beta$, &c.

The symbols are not in general commutative, $\alpha\beta$ not $= \beta\alpha$; but they are associative, $\alpha\beta.\gamma = \alpha.\beta\gamma$, each $= \alpha\beta\gamma$, which has thus a determinate signification.

[The associativeness of such symbols arises from the circumstance that the definitions of α, β, γ, . . determine the meanings of $\alpha\beta$, $\alpha\gamma$, &c.: if α, β, γ . . were quasi-quantitative symbols such as the quaternion imaginaries i, j, k, then $\alpha\beta$ and $\beta\gamma$ might have by definition values δ and ε such that $\alpha\beta.\gamma$ and $\alpha.\beta\gamma$ ($= \delta\gamma$ and $\alpha\varepsilon$ respectively) have unequal values].

Unity as a functional symbol denotes that the letters are unaltered, $1(x, y, z) = (x, y, z)$; whence $1\alpha = \alpha 1 = \alpha$.

The functional symbols *may* be substitutions; $\alpha(x, y, z) = (y, z, x)$, the same letters in a different order: substitutions can be represented by the notation $\alpha = \dfrac{yzx}{xyz}$, the substitution which changes xyz into yzx, or as products of cyclical substitutions, $\alpha = \dfrac{yzx\ wu}{xyz\ uw}$, $= (xyz)\ (uw)$, the product of the cyclical interchanges x into y, y into z, and z into x; and u into w, w into u.

A set of symbols $\alpha, \beta, \gamma \ldots$ such that the product $\alpha\beta$ of each two of them (in each order, $\alpha\beta$ or $\beta\alpha$,) is a symbol of the set, is a group. It is easily seen that 1 is a symbol of every group, and we may therefore give the definition in the form that a set of symbols, $1, \alpha, \beta, \gamma$ . . satisfying the foregoing condition is a group. When the number of the symbols (or terms) is $= n$, then the group is of the nth order; and each symbol α is such that $\alpha^n = 1$, so that a group of the order n is, in fact, a group of symbolical nth roots of unity.

A group is defined by means of the laws of combination of its symbols: for the statement of these we may either (by the introduction of powers and products) diminish as much as may be the number of independent functional symbols, or else, using distinct letters for the several terms of the group, employ a square diagram as presently mentioned.

Thus in the first mode, a group is $1, \beta, \beta^2, \alpha, \alpha\beta, \alpha\beta^2$ ($\alpha^2 = 1, \beta^3 = 1, \alpha\beta = \beta^2\alpha$); where observe that these conditions imply also $\alpha\beta^2 = \beta\alpha$:

Or in the second mode calling the same group $(1, \alpha, \beta, \gamma, \delta, \varepsilon)$, the laws of combination are given by the square diagram

	1	α	β	γ	δ	ε
1	1	α	β	γ	δ	ε
α	α	1	γ	β	ε	δ
β	β	ε	δ	α	1	γ
γ	γ	δ	ε	1	α	β
δ	δ	γ	1	ε	β	α
ε	ε	β	α	δ	γ	1

for the symbols $(1, \alpha, \beta, \gamma, \delta, \varepsilon)$ are in fact $= (1, \alpha, \beta, \alpha\beta, \beta^2, \alpha\beta^2)$.

The general problem is to find all the groups of a given order n; thus if $n = 2$, the only group is $1, \alpha$ ($\alpha^2 = 1$); $n = 3$, the only group is $1, \alpha, \alpha^2$ ($\alpha^3 = 1$); $n = 4$, the groups are $1, \alpha, \alpha^2, \alpha^3$ ($\alpha^4 = 1$), and $1, \alpha, \beta, \alpha\beta$ ($\alpha^2 = 1, \beta^2 = 1, \alpha\beta = \beta\alpha$);* $n = 6$, there are three groups, a group $1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5$

*If $n = 5$, the only group is $1, a, a^2, a^3, a^4$ ($a^5 = 1$). W. E. S.

($\alpha^6 = 1$) ; and two groups $1, \beta, \beta^2, \alpha, \alpha\beta, \alpha\beta^2$ ($\alpha^2 = 1.\ \beta^3 = 1$), viz: in the first of these $\alpha\beta = \beta\alpha$; while in the other of them (that mentioned above) we have $\alpha\beta = \beta^2\alpha$, $\alpha\beta^2 = \beta\alpha$.

But although the theory as above stated is a general one, including as a particular case the theory of substitutions, yet the general problem of finding all the groups of a given order n, is really identical with the apparently less general problem of finding all the groups of the same order n, which can be formed with the substitutions upon n letters; in fact, referring to the diagram, it appears that $1, \alpha, \beta, \gamma, \delta, \varepsilon$ may be regarded as substitutions performed upon the six letters $1, \alpha, \beta, \gamma, \delta, \varepsilon$, viz: 1 is the substitution unity which leaves the order unaltered, α the substitution which changes $1\alpha\beta\gamma\delta\varepsilon$ into $\alpha 1\gamma\beta\varepsilon\delta$, and so for $\beta, \gamma, \delta, \varepsilon$. This, however, does not in any wise show that the best or easiest mode of treating the general problem is thus to regard it as a problem of substitutions: and it seems clear that the better course is to consider the general problem in itself, and to deduce from it the theory of groups of substitutions.

CAMBRIDGE, *26th November, 1877.*

NOTE ON THE THEORY OF ELECTRIC ABSORPTION.

By H. A. Rowland.

In experimenting with Leyden jars, telegraph cables and condensers of other forms in which there is a solid dielectric, we observe that after complete discharge a portion of the charge reappears and forms what is known as the residual charge. This has generally been explained by supposing that a portion of the charge was conducted below the surface of the dielectric, and that this was afterwards conducted back again to its former position. But from the ordinary mathematical theory of the subject, no such consequence can be deduced, and we must conclude that this explanation is false. Maxwell, in his "Treatise on Electricity and Magnetism," vol. 2, chap. x, has shown that a substance composed of layers of different substances can have this property. But the theory of the whole subject does not yet seem to have been given.

Indeed, the general theory would involve us in very complicated mathematics, and our equations would have to apply to non-homogeneous, crystalline bodies in which Ohm's law was departed from and the specific inductive capacity was not constant; we should, moreover, have to take account of thermo-electric currents, electrolysis, and electro-magnetic induction. Hence in this paper I do not propose to do more than to slightly extend the subject beyond its present state and to give the general method of still further extending it.

Let us at first, then, take the case of an isotropic body in general, in which thermo-electric currents and electrolysis do not exist, and on and in which the changes of currents are so slow that we can omit electro-magnetic induction. The equations then become *

$$\frac{d}{dx}\left(\varkappa\frac{dV}{dx}\right)+\frac{d}{dy}\left(\varkappa\frac{dV}{dy}\right)+\frac{d}{dz}\left(\varkappa\frac{dV}{dz}\right)+4\pi\rho=0$$

$$\frac{d}{dx}\left(k\frac{dV}{dx}\right)+\frac{d}{dy}\left(k\frac{dV}{dy}\right)+\frac{d}{dz}\left(k\frac{dV}{dz}\right)-\frac{d\rho}{dt}=0$$

* Maxwell's Treatise, Art. 325.

in which $\varkappa$ is the specific inductive capacity of the substance, k the electric conductivity, V the potential, ρ the volume density of the electricity, and t the time.

The subtraction of one equation from the other gives

$$(1)\qquad \frac{dV}{dx}\frac{d}{dx}\left(\log\frac{k}{\varkappa}\right)+\frac{dV}{dy}\frac{d}{dy}\left(\log\frac{k}{\varkappa}\right)+\frac{dV}{dz}\frac{d}{dz}\left(\log\frac{k}{\varkappa}\right)-\frac{1}{k}\frac{d\rho}{dt}-\frac{4\pi\rho}{\varkappa}=0.$$

To introduce the condition that there shall be no electric absorption, we must observe that when that phenomenon exists, a charge of electricity appears at a point where there was no charge before; in other words, the relative distribution has been changed. Hence, if the relative distribution remains the same, no electric absorption can take place. Our condition is, then,

$$\frac{\rho}{\rho'}=c,$$

where c is independent of t, and ρ and ρ' are the densities at the points x, y, z, and x', y', z'. This gives

$$\frac{d}{dt}\left(\log\frac{\rho}{\rho'}\right)=0,$$

$$(2)\qquad\text{or}\qquad \frac{1}{\rho}\frac{d\rho}{dt}=\frac{d}{dt}\left(\log\frac{\rho}{\rho_0}\right)=-c,$$

where c is a function of t only and not of x, y, z, and ρ_0 is the value of ρ at the time $t=0$. As we have

$$\frac{1}{m}\frac{dV}{dn}\frac{dm}{dn}=\frac{dV}{dx}\frac{d}{dx}\left(\log\frac{k}{\varkappa}\right)+\frac{dV}{dy}\frac{d}{dy}\left(\log\frac{k}{\varkappa}\right)+\frac{dV}{dz}\frac{d}{dz}\left(\log\frac{k}{\varkappa}\right),$$

where $m=\frac{k}{\varkappa}$ and n is a line in the direction of the current at the given point, equation (1) becomes

$$\frac{1}{m}\frac{dV}{dn}\frac{dm}{dn}-\frac{1}{k}\frac{d\rho}{dt}-\frac{4\pi\rho}{\varkappa}=0,$$

From equation (2) $\qquad \rho=\rho_0\,\varepsilon^{-\int_0^t c\,dt},$

and hence

$$\frac{1}{m}\frac{dV}{dn}\frac{dm}{dn}+\rho_0\,\varepsilon^{\int_t^0 c\,dt}\left(\frac{c}{k}-\frac{4\pi}{\varkappa}\right)=0.$$

If we denote the strength of current at the point by S, we have

$$S = -k\frac{dV}{dn},$$

and

$$(3)\qquad \frac{1}{cm - 4\pi m^2}\,\frac{dm}{dn} = +\frac{\rho_0}{S}\,\varepsilon^{\int_t^0 c\,dt}\;;$$

this equation (3) gives the value of $\frac{k}{\varkappa} = m$ at all points of the body and at all times so that the phenomenon of electric absorption shall not take place As this equation makes m a function of x, y, z, S and t, the relation in general is entirely too complicated to ever apply to physical phenomena, without some limitation. Firstly then, as c is only an arbitrary function of t, we shall assume that it is constant;

$$\therefore\quad \frac{1}{cm - 4\pi m^2}\,\frac{dm}{dn} = +\frac{\rho_0\varepsilon^{-ct}}{S}\,.$$

The most important case is where m is a constant. Then

$$\frac{dm}{dn} = 0,$$

and

$$c = 4\pi m, \qquad S = S_0\,\varepsilon^{-ct}, \qquad \rho = \rho_0\,\varepsilon^{-ct}\,.$$

In this case, therefore, we see that both the electrification and the currents die away at the rate c. The case where Ohm's law is true and the specific inductive capacity is constant is included in this case, seeing that when k and $\varkappa$ are both constants their ratio, m, is constant. But it also includes the cases where k and $\varkappa$ are both the same functions of V, S, or x, y, z, seeing that their ratio, m, would be constant in this case also.

When m is not constant, the chances are very small against its satisfying equation (4).

Hence, we may in general conclude, that electric absorption will almost certainly take place unless the ratio of conductivity to the specific inductive capacity is constant throughout the body.

This ratio, m, may become a variable in several manners, as follows:

1st manner.—The body may not be homogeneous. This includes the case, which Maxwell has given, where the dielectric was composed of layers of different substances.

2d manner.—The body may not obey Ohm's law; in this case k would be variable.

3d manner.—The specific inductive capacity, $\varkappa$, may vary with the electric force.

It is to be noted that the cases of electric absorption which we observe are mostly those of condensers formed of two planes, or of one cylinder inside another, as in a telegraph cable. Our theory shows that different explanations can be given of these two cases.

The case of parallel plates does not admit of being explained, except on the supposition that m varies in the first manner above given, or in this manner in combination with the others, for we can only conceive of the conductivity and the specific inductive capacity as being functions of the ordinate or of the electric force. As the latter is constant for all points between the plates, m would still be constant although it were a function of the electric force, and thus electric absorption would not take place.

We may then conclude that in the case of parallel plates, omitting explanations based on electrolysis or thermo-electric currents, the only explanation that we can give at present is that which depends on the non-homogeneity of the body, and is the case which Maxwell has given in the form of two different materials. Our equations show that the form of layers is not necessary, but that any departure from homogeneity is sufficient. It is to be noted that the homogeneity, which we speak of, is electrical homogeneity, and that a mass of crystals with their axes in different directions would evidently not be electrically homogeneous and would thus possess the property in question. In the case of glass it is very possible that this may be the case and it would certainly be so for ice or any other crystalline substance which had been melted and cooled.

In the case of hard india rubber, the black color is due to the particles of carbon, and as other materials are incorporated into it during the process of manufacture, it is certainly not electrically homogeneous.

As to the ordinary explanation that the electricity penetrates a little below the surface and then reappears again to form the residual charge, we see that it is in general entirely false. We could, indeed, form a condenser in which the surface of the dielectric would be a better conductor than the interior and which would act thus. But in general, the theory shows that the action takes place throughout the mass of the dielectric, where that is of a

fine grained structure and apparently homogeneous, as in the case of glass, and consists of a polarization of every part of the dielectric.

To consider more fully the case of a condenser made of parallel plates, let us resume our original equations. Without much loss of generality we can assume a laminated structure of the substance in the direction of the plane YZ, so that m and V will be only functions of the ordinate x. Our equations then become

$$\frac{d}{dx}\left(\varkappa\frac{dV}{dx}\right)+4\pi\rho=0,$$

$$\frac{d}{dx}\left(k\frac{dV}{dx}\right)-\frac{d\rho}{dt}=0.$$

Eliminating ρ we find

$$\frac{1}{4\pi}\frac{d}{dt}\frac{d}{dx}\left(\varkappa\frac{dV}{dx}\right)+\frac{d}{dx}\left(k\frac{dV}{dx}\right)=0.$$

Now let us make $p=\varkappa\dfrac{dV}{dx}$, and as t and x are independent, we find on integration,

$$\frac{d}{dt}(p-p_0)+4\pi(pm-p_0m_0)=0,$$

where p_0 is the value of p for some initial value of x, say at the surface of the condenser, and is an arbitrary function of t, seeing that we may vary the charge at the surface of the body in any arbitrary manner. This equation establishes p as a function of m and t only, and as we have

$$\rho=-\frac{1}{4\pi}\frac{dp}{dx},$$

ρ will also be a function of these only.

Let us now suppose that at the time $t=0$, the condenser is charged, having had no charge before, and let us also suppose that the different strata of the dielectric are infinitely thin and are placed in the same order and are of the same thickness at every part of the substance, so that a finite portion of the substance will have the same properties at every part.

In this case m will be a periodic function of x, returning to the same value again and again. As ρ is a function of this and of t only, at a given time t, it must return again and again to the same value as we pass through the substance, indicating a uniform polarized structure throughout the body.

This conclusion would have been the same had we not assumed a laminated structure of the dielectric. In all other cases, except that of two planes, electric absorption can take place, as we have before remarked, even in perfectly homogeneous bodies, provided that Ohm's law is departed from or that the electric induction is not proportional to the electric force, as well as in non-homogeneous bodies. But where the body is thus homogeneous, electric absorption is not due to a uniform polarization, but to distinct regions of positive and negative electrification.

In the whole of the investigation thus far we have sought for the means of explaining the phenomenon solely by means of the known laws of electric induction and conduction. But many of the phenomena of electric absorption indicate electrolytic action, and it is possible that in many cases this is the cause of the phenomenon. The only object of this note is to partially generalize Maxwell's explanation, leaving the electrolytic and other theories for the future.

ESPOSIZIONE DEL METODO DEI MINIMI QUADRATI.

PER ANNIBALE FERRERO, *Tenente Colonnello di Stato Maggiore, ec. Firenze, 1876.*

BY CHARLES S. PEIRCE, *New York.*

RECENT discussions in this country, of the literature of the method of Least Squares, have passed by without mention the views of the accomplished chief of the geodetical division of the Italian Survey, as set forth in the work above cited, which was first published, in part, in 1871. The subject is here, for the first time, in my opinion, set upon its true and simple basis; at all events the view here taken is far more worthy of attention than most of the proposed proofs of the method.

Lieut. Col. Ferrero begins by considering the principles of the arithmetical mean. A quantity having been directly observed, a number of times, independently, and under like circumstances, the value which might be inferred from the observations is, in the first place, a symmetrical function of the observed quantities; for, if the observations are independent, the order of their occurrences is of no consequence, and the circumstances under which they are taken, differ in no assignable respect, except that of being taken at different times. In the second place, the value inferred must be such a function of the values observed, that when the latter are all equal, the former reduces to this common value. The author calls functions having these two properties, (1st, that of being symmetrical with respect to all the variables, and 2d, that of reducing to the common value of the variables when these are all equal,) ***means***. There is a whole class of functions of this sort, such as the arithmetic mean, the geometrical mean, the arithmetic-geometrical mean of Gauss, the quadratic mean,* and many others instanced in the text. It is shown, without difficulty, that these means are continuous functions, and that their value is intermediate between the extreme values of the different variables, when the latter do not differ greatly.

Let o', o'', o''', etc. denote the values given by the observations. Let n denote the number of the observations; let p denote the arithmetical mean;

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*This seems the appropriate name for $\sqrt{\frac{[x^2]}{n}}$.

and let x', x'', x''', etc. denote the excess of the observed values over the arithmetical mean. Then write

$$V = f\,(o', o'', o''', \text{etc.})$$

for any mean of the observations. Develop this function according to powers of x', x'', x''', etc. We have

$$V = f\,(p + x', p + x'', p + x''', \text{etc.})$$

$$= f\,(p, p, p, \text{etc.}) + \frac{dV}{dp}\,(x' + x'' + x''' + \text{etc.}) + \Delta\,;$$

where Δ denotes the terms of higher orders.

Since $$x' + x'' + x''' + \text{etc.} = 0,$$

and $$f\,(p, p, p, \text{etc.}) = p,$$

this reduces to

$$V = p + \Delta.$$

In considering the value of Δ, we may limit ourselves to terms of the second order. As the partial differentials of any species and order, relatively to o', o'', o''', etc. all become equal when x', x'', x''', etc. vanish, we may write

$$\frac{d^2V}{do'^2} = \frac{d^2V}{do''^2} = \frac{d^2V}{do'''^2} = \text{etc.} = \beta$$

$$\frac{d^2V}{do'.do''} = \frac{d^2V}{do''.do'''} = \text{etc.} = \gamma$$

then

$$\Delta = \tfrac{1}{2}\,\beta\,(x'^2 + x''^2 + x'''^2 + \text{etc.}) + \gamma\,(x'x'' + x'x''' + \text{etc.})\,.$$

But the square of $[x] = 0$, gives

$$\Sigma\, x\, x' = -\tfrac{1}{2}\,[x^2]\,,$$

so that

$$\Delta = \frac{\beta - \gamma}{2}\,[x^2] = k\,\frac{[x^2]}{n}\,,$$

where k is a quantity which does not increase indefinitely with n. Now, when the observations are good, $\frac{[x^2]}{n}$ is not large, and, therefore, in such a case no mean will differ very much from the arithmetical mean. The latter, being the simplest to deal with, may therefore be used without great disadvantage. Such is, according to Colonel Ferrero, the utmost defence of the principle which can be made to cover all the cases in which it is usual to employ the method; and all further defence of it is more or less limited in its application.

In very many cases, however, it is easy to see that either in regard to the quantity directly observed, or in regard to some function of it, the zero of the scale of measurement, and the unit of the same scale, are both arbitrary. For instance, in photometric observations, this is true of the logarithm of the light. In such cases, considering such function to be the observed quantity, we have there two principles, first proposed, in connection with a really superfluous third one, by Schiaparelli.

1st. The mean to be adopted must be such that if each observed value is multiplied by any constant, the result is increased in the same ratio.

2d. The mean to be adopted must be one which is increased by a constant o, when each observed value is increased by the same constant.

Our author's treatment of these principles is exceedingly neat. Using the same notation as above, write

$$V = p + A_2 + A_3 \ldots + A_n \ldots$$

where A_n is the sum of the terms of the order n in x', x'', x''', etc. The general term A_n is, therefore, of the form $A_n = \alpha \Sigma x'^n + \beta \Sigma x'^{n-1} x'' + \gamma \Sigma x'^{n-2} x''^2 \ldots + \zeta \Sigma x'^r x''^s x'''^t \ldots$ where Σ expresses the symmetrical sum of similar terms. In the general term $r + s + t +$ etc. $= n$. Since ζ is evidently a function of p, we may put $\zeta = \phi(p)$, and it remains to find the form of this function. Multiplying every o by c, p is changed to cp, x to cx, and the general term $\zeta \Sigma x'^r x''^s x'''^t$ etc. $= \phi(p) \Sigma x'^r x''^s x'''^t$ etc. is changed to $\phi(cp) c^n \Sigma x'^r x''^s x'''^t$ etc. Since, therefore, V is changed to cV, we have $\phi(cp) c^n = \phi(p) c$. Putting $p = 1$, $\phi(c) = \dfrac{\phi(1)}{c^{n-1}}$. Denoting this numerator by ξ_1, the general term becomes

$$A_n = \frac{1}{p^{n-1}} \left[\alpha_1 \Sigma x'^n + \ldots + \xi_1 \Sigma x'^r x''^s x'''^t \ldots + \ldots \right],$$

where α, ξ, etc., are numerical coëfficients independent of p. From this circumstance it follows that the quantity in square brackets, which may be called A'_n, does not change when the same constant quantity k is added to all the observed quantities o', o'', o''', etc.; for such an addition only increases p by this same constant, and leaves x', x'', x''', etc., unchanged. Thus the mean in question, which may now be written

$$V = p + \frac{A'_2}{p} + \frac{A'_3}{p^2} + \text{etc.},$$

becomes, in consequence of such an addition,

$$V_k = p + k + \frac{A'_2}{p+k} + \frac{A'_3}{(p+k)^2} + \text{etc.}$$

But by principle No. 2, it becomes,

$$V_k = p + k + \frac{A'_2}{p} + \frac{A'_3}{p^2} = \text{etc.}$$

So that, $A'_2 = A'_3 = \text{etc.} = 0$, and we have

$$V = p,$$

or the arithmetical mean is the only one which conforms to the given conditions.

Another still more special case, is that contemplated by the demonstrations of Laplace, Poisson, Hagen, Crofton, etc. It is treated by our author, but need not be considered in this notice.

It may be of interest to see how Colonel Ferrero is able, without basing least squares expressly upon the theory of probabilities, to derive the formula for finding mean error. Using always the same notation, he terms

$$m = \sqrt{\frac{[x^2]}{n}}$$

the *mean residual* of the observations.

Suppose, then, that there be an indefinitely great *series of series* of observations of the same quantity, each lesser series consisting of n observations, and each having the same mean residual. Then, there being an infinite number of such series, the mean of their mean results may be taken as the true value, by definition. For the ultimate result of indefinitely continued observation is all that we aim at in sciences of observation. Then the number of the lesser series being q, the result will be

$$V = \frac{[p]}{q}.$$

Adopt the notation

$$\delta = p - V \quad \delta_1 = p_1 - V \quad \delta_2 = p_2 - V, \text{ etc.},$$

then δ, δ_1, δ_2, etc., are the true errors of p, p_1, p_2, etc. Let y'_0, y''_0, y'''_0, etc. be the true errors of the first series of observations, y'_1, y''_1, y'''_1 etc. those of the second series, and so for the others. We have, then, $y = o - V = o - p + \delta = x + \delta$.

Squaring and summing for the nq values of y, we have

$$\Sigma y^2 = \Sigma x^2 + \Sigma\delta^2 + 2\Sigma x\Sigma\delta$$

or, since $\qquad \Sigma x = 0, \qquad \text{and} \qquad \Sigma\delta = 0,$

$$\Sigma y^2 = \Sigma x^2 + \Sigma\delta^2.$$

Now if η be the quadratic mean of the error of p, we have $\Sigma\delta^2 = nq\eta^2$, and

$$\Sigma y^2 = nqm^2 + nq\eta^2,$$

or the mean error μ of an observation is given by

$$\mu^2 = \frac{\Sigma y^2}{nq} = m^2 + \eta^2.$$

But it is easily shown (from the equality of positive and negative errors) that

$$\eta^2 = \frac{\mu^2}{n}$$

whence

$$\mu = \sqrt{\frac{[x^2]}{n-1}}\,.$$

With regard to the mode of passing from the principle of the arithmetical mean to the general method of least squares, the best way seems to be first to prove that the solution of the equations

$$\begin{gathered} a_1 x = n_1 \\ a_2 x = n_2 \\ \text{etc.,} \end{gathered}$$

is $x = \frac{[an]}{[a^2]}$. This is easy, after the rule for the error of a mean is established. Then, having given the equations

$$\begin{gathered} a_1 x + b_1 y + c_1 z + \text{etc.} = n_1 \\ a_2 x + b_2 y + c_2 z + \text{etc.} = n_2\,; \end{gathered}$$

first, consider these as similar to the equations just given; thus,

$$\begin{gathered} a_1 x = n_1 - b_1 y - c_1 z - \text{etc.,} \\ a_2 x = n_2 - b_2 y - c_2 z - \text{etc.,} \\ \text{etc.,} \end{gathered}$$

whence we obtain the first normal equation,

$$x = \frac{[an_1] - [ab]\,y - [ac]\,z - \text{etc.}}{[a^2]}$$

and the others in a similar way.

The treatise of Colonel Ferrero may be recommended to those desirous of having a thorough practical acquaintance with the method, as decidedly the best and clearest on the subject.

ON AN APPLICATION OF THE NEW ATOMIC THEORY TO THE GRAPHICAL REPRESENTATION OF THE INVARIANTS AND COVARIANTS OF BINARY QUANTICS,—WITH THREE APPENDICES.

By J. J. Sylvester.

By the *new* Atomic Theory I mean that sublime invention of Kekulé which stands to the *old* in a somewhat similar relation as the Astronomy of Kepler to Ptolemy's, or the System of Nature of Darwin to that of Linnæus;—like the latter it lies outside of the immediate sphere of energetics, basing its laws on pure relations of form, and like the former as perfected by Newton, these laws admit of exact arithmetical definitions.

Casting about, as I lay awake in bed one night, to discover some means of conveying an intelligible conception of the objects of modern algebra to a mixed society, mainly composed of physicists, chemists and biologists, interspersed only with a few mathematicians, to which I stood engaged to give some account of my recent researches in this subject of my predilection, and impressed as I had long been with a feeling of affinity if not identity of object between the inquiry into compound radicals and the search for "Grundformen" or irreducible invariants, I was agreeably surprised to find, of a sudden, distinctly pictured on my mental retina a chemico-graphical image serving to embody and illustrate the relations of these derived algebraical forms to their primitives and to each other which would perfectly accomplish the object I had in view, as I will now proceed to explain.

To those unacquainted with the laws of atomicity I recommend Dr. Frankland's Lecture Notes for Chemical Students, vols. 1 and 2, London (Van Voorst), a perfect storehouse of information on the subject arranged in the most handy order and put together and explained with true scientific accuracy and precision. On the algebraical side of the subject my readers may consult Salmon's "Lessons on Higher Algebra," Clebsch's "Binären Formen" or Faà de Bruno's treatise more elementary than the former, "Sur les formes binaires" (Turin, 1876). I propose also to run a course of articles on the Invariative Theory, beginning from the beginning, through the pages of this

Journal, from my own particular point of view which will be found, I hope, considerably to simplify the subject.

Any binary quantic may be denoted by a single letter with a number attached corresponding to its degree, and may therefore be adumbrated by a chemical symbol with corresponding *valence.* Thus hydrogen, chlorine, bromine, or potassium will serve to denote so many distinct binary linear forms; oxygen, zinc, magnesium, etc., binary quadrics; boron, gold, thallium, cubics; carbon, lead, silicon, tin, quartics; nitrogen, phosphorus, arsenic, antimony, etc., quintics; sulphur, iron, cobalt, nickel, etc., sextics. The sixth appears to be the highest degree of valency at present recognizable in natural substances.

The factors of any algebraical form may be regarded as in some sense the analogues of the rays of atomicity in the equi-valent chemical atom— these rays being what Dr. Frankland, according to his nomenclature, would have to designate as free bonds; such rays between two consecutive atoms in a molecule are conceived as blending in some manner so as to represent some unknown kind of special relation existing between them; they may then with propriety be called bonds or lines of connexion.

An invariant of a form or system of algebraical forms must thus represent a saturated system of atoms in which the rays of all the atoms are connected into bonds. Thus, ex. gr. O_2 (oxygen combined with itself) will represent a quadratic invariant of a quadric. Its graph is seen in Fig. 1, (a). Potash, a combination of potassium, oxygen and hydrogen, having for its graph that of Fig. 2, will represent the invariant to a system of one quadratic and two linear forms which is linear in each set of coëfficients. This is in fact the *Connective* between the given quadratic and another obtained by taking the product of the two linear forms. Phosphorus and arsenic are quinquivalent, but form "tetratomic molecules." An isolated element of phosphorus may possibly, therefore, be represented by the graph of Fig. 3, which will correspond, if the figure is indecomposable (which requires examination to determine), to the quart-invariant of a quintic, and the same for arsenic. So too the graph to nitric anhydride (Fig. 4) may possibly serve to express the resultant of a binary quadric and quintic, or this blended with any other invariant of the system included under the same type [10: 5, 2; 2, 5].*

* 10 is the weight; 5, 2 the degree and order in the coëfficients of the quintic; 2, 5 the degree and order in the coëfficients of the quadric. See p. 67.

And in general, the Jacobian to any two quantics will be completely expressed by their two corresponding atoms connected by a pair of bonds. Nitric acid has for its graph that of Fig. 5. This will correspond to an invariant of a quintic, quartic and linear form of the first order in the coëfficients of each extreme and of the third order in those of the middle form. Such an invariant as is well known, (by virtue of a general principle about to be stated) is, in substance, the same thing as a lineo-cubic linear covariant of a quintic and quadric. The general arithmetical rule (also hereafter to be set forth) for determining the number of asyzygetic derivatives of a given type, enables us to see that such a covariant exists and is monadelphic. It may readily be obtained by making the given quintic (after substituting $\frac{d}{dy}$ and $-\frac{d}{dx}$ for x and y respectively) operate on the cube of the given quadratic.

The general principle above referred to, which is extremely easily proved from the partial differential equation, (but which I believe I was the first to enunciate) is that every covariant of one quantic or several simultaneous quantics may be transformed into an invariant of the same quantic or set of quantics enlarged by the addition thereto of one additional linear form; the degree in the variables becoming replaced by the order in the new set of coëfficients, and the orders in the original sets of coëfficients remaining unchanged.

Thus, covariants might altogether be dispensed with and invariants alone made the object of study. But algebraists have found and will continue to find it more convenient to dispense with the additional linear form and to retain in use covariants as well as invariants. With me, covariants are to be regarded as simple emanations, so to say, from differentiants which are functions of the coëfficients alone, and of which invariants are merely a particular species satisfying a certain condition of maximum; this is why the properties of invariants can with difficulty be made out so long as they are studied alone; it was only by contemplating the whole group of differentiants simultaneously, that I was enabled, after a suspense of more than a quarter of a century, to set on an irrefragible basis Professor Cayley's fundamental arithmetical theorem for calculating the number of asyzygetic invariants and covariants to a given quantic, and also the more general theorem which I have shown applies to a system of quantics.*

* The demonstration is given in a paper inserted in the Philosophical Magazine for March of this year.

I will here give this rule, as it may be useful to us in the sequel. First, for a single quantic.— Let i be its degree, j the order of any covariant, w its weight (i. e. the weight of its root-differentiant). Then we may call its type $[w : i, j]$. Now let us, in general, employ $(m : i, j)$ to signify *the number of ways* in which m can be made up with i parts of which each is either 0, 1, 2, 3 etc. up to i, and let us use the symbol $\Delta\,(m : i, j)$ to denote $(m : i, j) — (\overline{m-1} : i, j)$; then $\Delta\,(w : i, j)$ is the number of arbitrary numerical parameters in the most general covariant or invariant answering to the type $[w : i, j]$. It is a known theorem in partitions of numbers that $(m : i, j) = (m : j, i)$, from which it follows that the number of arbitrary parameters remains unaltered when the degree of the primitive and the order of the derivative are interchanged. It is sometimes more convenient to use the degree of the derivative in lieu of the weight to express its type; let then ε be the degree, so that $\varepsilon = 2ij — w$; then I shall employ, when desirable, $[i, j : \varepsilon]$ to signify the same thing as $[w : i, j]$. If there be several quantics, the type may be expressed in like manner by $[w : i, j;\ i', j';$ etc.$]$, or by $[i, j;\ i', j';$ etc.$: \varepsilon]$. The rule for finding the number of independent parameters, or the most general covariant or invariant corresponding to either of these types, then becomes as follows. Let $(m : i, j;\ i', j';$ etc.$)$ denote the number of ways in which m can be made up of j elements each comprised between 0 and i, combined with j' elements each comprised between 0 and i', and so on, and let $\Delta\,(m : i, j;\ i', j;$ etc.$)$ denote $(m : i, j;\ i', j';$ etc.$)$ — $(m-1 : i, j;\ i', j';$ etc.$)$. The number of parameters in question is $\Delta\,(w : i, j;\ i', j';$ etc.$)$ and I may observe that the value of Δ remains unaltered when *any one* i is interchanged with the corresponding j, and consequently when any number of i's are interchanged, each respectively with its corresponding j. This theorem of reciprocity for a single quantic is due to M. Hermite. The above statement, applicable to a quantic system, constitutes a notable and important generalization of it. In Note D to Appendix 2, it will be shown that this theorem still further generalized by employing the method of Emanation (virtually the same thing as Regnault's law of substitution) admits of the following simple chemico-algebraical statement. *In an algebraical compound (in an algebraical sense) m n-valent atoms may be replaced by n m-valent ones.* But it should be observed that this replacement involves an entire reconstruction of the representative graph and conveys the notion of respondence or contraposition rather than similarity of type. (See Appendix 2.)

It may be well here (as it will be useful in the sequel) to say a few words more on these differentiants in their relation to covariants. Every covariant may be regarded as arising from either of two differentiants, as from a root. One, the coëfficient of the highest power of x, is called a differentiant in x; the other, the coëfficient of the highest power of y, a differentiant in y. It is not, for ordinary purposes such as present themselves in this study, requisite to consider more than one of these at a time, and for greater brevity it will be understood that, unless I give notice to the contrary, a *differentiant* will always be understood to mean one in x. I shall also suppose, when dealing with a single binary quantic, that the successive coëfficients beginning with the highest power of x, are $a, b, c, \ldots . h, k, l$ multiplied successively by the binomial coëfficients proper to the degree of the form. A differentiant, D, may then be defined as a rational integer function of the coëfficients of equal weight in all its terms in respect to either variable subject to satisfy the equation

$$\left(a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}+\ldots .\right)D=0.$$

An invariant again may be regarded as a rational integer isobaric function of the coëfficients which is a differentiant both in regard to x and y, but it may be best defined as a differentiant (meaning in one of the variables as x,) to a given form or form-system whose weight (in respect of the selected variable) is the greatest possible that its order in the coëfficients admits of. [The doubleness of the character and the symmetry, direct or skew, of a differentiant satisfying this condition of maximum then become matter of deduction from the definition.] To each covariant corresponds but one differentiant (in a given variable) and *vice versa*, to each differentiant will correspond only one covariant. In fact, D being the differentiant in x, the covariant taking its rise in D is

$$Dx^{\varepsilon}+\Omega\cdot Dx^{\varepsilon-1}\cdot y+\frac{1}{1\cdot 2}(\Omega\cdot)^2Dx^{\varepsilon-2}y^2+\ldots .,$$

where $\Omega\cdot$ represents the operator,

$\left(l\frac{d}{dk}+2k\frac{d}{dh}+3h\frac{d}{dg}+\ldots .\right)$ if D belongs to a simple quantic, and $\Sigma\left(l\frac{d}{dk}+2k\frac{d}{dh}+\ldots .\right)$ if it belongs to a quantic system, and where ε is $ij-2w$ for a single quantic, and $\Sigma ij-2w$ for a quantic system, i representing the degree of any one form in the variables, j the order of the

differentiant in the corresponding set of coëfficients, and w the weight of the differentiant. As ε can never become negative, we see that the maximum value of w, when each i and its corresponding j is given, will be $\frac{ij}{2}$ for one form, and $\frac{1}{2}\Sigma ij$ for a form system. By the weight of any covariant I shall understand the weight of the differentiant in which it may be regarded as originating. Precisely as algebraists find their advantage in using covariants when invariants alone might be made to suffice, chemists find theirs in the use of organic or inorganic compound radicals, as unsaturated forms capable of becoming saturated by the addition of the right number of monad elements to the unsatisfied atoms, i. e. those through which a sufficient number of *bonds* do not pass to exhaust their valency. Thus ex. gr., Hydroxyl H — O — is the linear covariant of the quadratic form oxygen, and the linear form hydrogen; this, combined with the linear form potassium, expresses the invariant potash denoted by $H — O — K$.

As the free valence of a single atom corresponds to the degree of a single quantic, so the free valence of a molecule formed by an aggregate of atoms will express the degree of the corresponding covariant. Let us understand by the *toti-valence* of a molecule the sum of the absolute valences of the separate atoms of which it is composed. This toti-valence will obviously correspond to the sum, Σij, above mentioned. Since every bond or connecting line in the graph passes through two atoms, this toti-valence must be equal to the free valence of the molecules increased by twice the number of bonds; but Σij is the toti-valence, and ε (the degree of the covariant) is the number of unsatisfied bonds, and we have already stated in effect that ε increased by twice the weight of the root differentiant (which for brevity we call the weight of the covariant) is equal to Σij; hence the weight of a covariant (meaning that of its root differentiant), represented by any chemicograph, is the number of bonds or connecting lines between the atoms.

Let us consider an invariant or a covariant belonging to a type containing only one numerical parameter, which I shall call a monadelphic form.* Then this is either decomposable into factors or not; in the former case it may be termed composite, in the latter case prime. When prime its graph will also

* The type itself may also be termed a monadelphic type: so I shall speak when necessary of diadelphic, triadelphic, etc. types and designate any forms contained under such types as diadelphic, triadelphic, etc. forms. A family comprising many brothers, or any member of such a family, may each without doing violence to the laws or usage of language be termed *polyadelphic*.

be prime, when composite its graph will be composite in a sense which will be made more clear by one or two examples Let us take as a first example a graph composed of four triadic atoms of the same name, as in Fig. 6, where each atom, for instance, represents boron and in ordinary chemical symbolism would be denoted by the same letter B, but where for facility of reference I use four different letters to mark the positions of the several atoms. This corresponds to a covariant of a cubic for which the complete type, if we use the weight or number of bonds, is [4 : 3, 4], or, if we use the free valency, is [3, 4 : 4]. Now for a cubic the fundamental types, expressed in terms of the order and degree alone, omitting the constant number 3, which refers to the given degree, are

$$\begin{array}{c} 1 \,.\, 3 \\ 4 \,.\, 0 \\ 2 \,.\, 2 \\ 3 \,.\, 3\,. \end{array}$$

Consequently, there is but one covariant corresponding to the given graph, and that is the product of the primitive by the covariant whose order and degree are each 3, the well known skew covariant $(a, b, c, d \between x, y)^3$ whose root or base is the differentiant $ad^2 - 3abc + 2b^3$.

It must be well understood that the bonds are not rigid, but capable of being curved or bent into any desired form. In this case the mode of decomposition is self-evident; for the skew covariant is represented by the triangle of Fig. 7, and we have only to draw out the elastic bond AC into the position ADC and place the atom D anywhere upon it to obtain the given graph. On the contrary the skew covariant itself is indecomposable and its graph ABC is obviously so too. Now let us consider the graph of Fig. 8. If the atoms at the angles are all triadic, there is no free valency, and the figure represents the invariant to a cubic form corresponding to 4 . 0 in the above table. It will be found, on trial, impossible to decompose it. But now suppose the atoms to be tetradic, the graph will represent a covariant of the fourth order and of the fourth degree to a quartic, each atom having one degree of valency unsatisfied. The fundamental derivatives of a quartic, of which all others are algebraical combinations, are represented in the following table of order and degree

$$\begin{array}{c} 1 \,.\, 4 \\ 2 \,.\, 0 \\ 3 \,.\, 0 \\ 2 \,.\, 4 \\ 3 \,.\, 3\,. \end{array}$$

The complete covariant answering to the graph will therefore be $\lambda U + \mu V$, where, λ, μ being arbitrary numbers, U is the product of the primitive (1.4) by the cubinvariant 3.0, and V the product of the Hessian 2.4 by the quadrinvariant 2.0. Since, on making either $\lambda = 0$ or $\mu = 0$ the covariant breaks up and in two different ways into factors, we ought to expect that the graph should be capable of two corresponding modes of decomposition, and such we shall easily see is the case. For 1°, the invariant 3.0 may be represented by the graph of Fig. 9. Now imagine the three points E, F, G to come together and blend at D and at D place a fourth atom. The given graph is thus recovered. Observe that this could not be done for the case of triads (corresponding to a cubic form) because, in the figure last referred to, the valence at each atom A, B, C is quadrivalent. Next, for the decomposition corresponding to the case of $\lambda = 0$ where the covariant breaks up into 2.0 multiplied by 2.4, the decomposition will be more easily followed by considering the graph to be pulled out into the form seen in Fig. 10. We may conceive this as the superposition of two carbon graphs, one in which the carbon atoms are at A and B connected by the *four* bonds AB, ACB, BDA, $ACDB$ denoting the quadrinvariant, and another in which the carbon atoms C, D are connected by the *two* bonds CAD, CBD, leaving two degrees of valence free at each atom and thus representing the quadro-quart-invariant or Hessian of the primitive.

I will now pass to the very interesting case which corresponds to one of the proposed graphs for benzole (or rather for the compound radical obtained by striking off its hydrogen atoms), a sextivalent hexad molecule of carbon—not the one proposed by Kekulé and which I believe still commands the general assent of chemists, but that suggested by Ladenburg* and put by him under the form of a wedge or prism. As, however, the question is one purely of colligation or linkage in the abstract, it is sufficiently described as a hexagon in which the three pairs of opposite angles are joined, or, if we please, as two triangles in which each angle of one is connected with a corresponding angle of the other. In regard of the atomicity theory, all these modes of colligation are identical, and the supposition that there is any real difference between them, or that figures in space are distinguishable from figures in a plane (as I heard suggested might be the case by a high authority at a meeting of the

* *Berichte der deutschen chemischen Gesellschaft, 1869, 141.* I am indebted for this reference to my able colleague, Professor Ira Remsen.

British Association for the Advancement of Science, where I happened to be present), is a departure from the cautious philosophical views embodied in the theory as it came from the hands of its illustrious authors and continued to be maintained by their sober-minded successors and coadjutors, and affords an instructive instance of the tendency of the human mind to the worship, as if of self-subsistent realities, of the symbols of its own creation.

The order (or number of atoms) being 6 and the unexhausted valences (one at each atom) also 6, we must turn to our table of fundamental derivatives to the quartic and shall find that the combination 6.6 is not amongst them, but that it can be obtained, and in only one way, by composition of the combinations therein contained. It is, in fact, the product of the cubic invariant 3.0 by the skew covariant 3.6, which has the very same *root* $a^2d - 3abc + 2b^3$ as the skew covariant to the cubic and accordingly has the same graph, namely a simple triangle. (It may be well to remark here incidentally, that it follows as an immediate consequence from the conditioning partial differential equation, that a root-differentiant to any quantic or system of quantics of given degree or degrees remains such to every other system in which one or more of those degrees is augmented.) On the other hand the cubic invariant has for its graph a triangle in which each line is doubled or looped. I shall show that Ladenburg's graph for the radical to benzole may be obtained by the superposition of these two forms. Let $ABC\gamma\beta\alpha$ represent a sextivalent tetradic hexad (Fig. 11); ABC, with the three loops $A\alpha\gamma C$, $C\gamma\beta B$, $B\beta\alpha A$, will represent a saturated triple atom of carbon, or the cubinvariant of a binary quartic. Again, $\alpha\gamma\beta$ taken alone will represent a sextivalent compound atom, or the fundamental skew covariant of the quartic, and the superposition of the two figures obviously gives the graph as it stands.

Another form of the product of the same two graphs would be a triangle inscribed in another, as in Fig. 12. Here $\alpha\beta\gamma$, as before, is the sexivalent molecule and ABC with the additional bonds $A\beta C$, $B\gamma A$, $C\alpha B$, the saturated one.

A simple hexagon of triadic atoms (Fig. 13) being sextivalent will serve to represent a derivative from a cubic of the sixth order and sixth degree. Such a covariant, in its most general form, will contain two parameters and be represented by $\lambda U^3 + \mu V^2$ where U is the Hessian 2.2 and V the skew cube covariant 3.3, and it is easy to see that this figure may be decomposed either into 3 bivalent, or 2 trivalent graphs. Thus AB, CD, EF, with the additional

bonds $BCDEFA$, $DEFABC$, $FABCDE$, will represent the former; two atom groups such as A, C, E (with the bonds ABC, $AFEDC$, CDE, $CBAFE$, EFA, $EDCBA$) and B, D, F (with the bonds BCD, $BAFED$, DEF, $DCBAF$, FAB, $FEDCB$) the other. The first method of regarding the hexagon as a combination of three dyads may perhaps be admitted to throw some light on what Dr. Frankland styles the two distinct molecular weights of sulphur. When two atoms of sulphur, regarded as bivalent, are combined by two loops, we have a representation of an isolated element of it as "a diatomic molecule." When three of these letters, regarded now as submolecules, are combined, or multiplied together into the hexagon, we have a representation of the isolated element as "a hexatomic molecule." More generally, let μ be the number of solutions of the equation in positive integers $2x + 3y = m$, then μ arbitrary parameters will enter into the most general representation of a covariant to a cubic of the order m in the coëfficients and the degree m in the variables. Its graph will be a simple polygon of m sides and this will be capable of being decomposed, in μ essentially distinct ways, into elementary graphs consisting either, of binary groups or, ternary groups exclusively or, the two sorts of groups intermixed.

It may be easily shown (see Appendix 3) that every covariant of a binary form multiplied by a suitable power of its primitive, is capable of being represented by a rational integer function of covariants consisting, in addition to the primitive, of covariants exclusively of the second and third orders in the coëfficients. I have already given an example of the mode in which a graph may be augmented by an additional atom corresponding to the multiplication of a covariant by the primitive.

The important proposition above referred to (given in Clebsch's Binären Formen) amounts then to affirming that any homogeneous graph augmented by a suitable number of atoms of the same, may be decomposed, in one or more ways, into bilooped dyads and single-sided triangles. Such a proposition ought to admit of graphical proof. The theorem has considerable graphical importance because it enables us, in some cases at least, to discriminate the true from the spurious graphs, or as we might say, pseudographs, representing a given type. Thus, it serves to show that Fig. 14 and not Fig. 15 is the graph to the discriminant of a cubic; for, in accordance with Clebsch's theorem, this discriminant, viz:

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd,$$

multiplied by a^2 becomes equal to the square of $a^2d - 3abc + 2b^3$, together with four times the cube of $(ac - b^2)^3$, and consequently its graph, after combination with two additional points, should be decomposable, at will, into 3 double looped lines, or into 2 single-lined triangles, which is the case with Fig. 14, inasmuch as its combination with two points gives rise to a simple hexagon, but not with the second.

If we call the apices of the two figures, 14, 15, a, b, c, d, the true graph (on substituting negative signs for bonds and prefixing a sign of summation) reads as

$$\Sigma\,(a-b)^2(c-d)^2(a-c)(b-d)\,,$$

which is the cubinvariant of the quartic whose roots are a, b, c, d, so that a graph to an invariant of the type [3, 4: 0] gives the algebraical expression in terms of the roots of an invariant of the reciprocal type [4, 3 : 0]. On the other hand, the pseudograph treated in the same way reads as

$$\Sigma\,(a-b)(b-c)(c-d)(d-a)(a-c)(b-d)\,,$$

the value of which is zero; a similar remark may probably be found to be true of reciprocal graphs of invariants in general. This is abundantly confirmed by subsequent investigation; see remarks at end of Appendix 1.

So again, if we take the graph of Fig. 42, which represents an invariant to the type [3, 2 ; 1, 2 : 0], it reads off into

$$\Sigma\,(B_1-B_2)^2(B_1-H_1)(B_2-H_2)\,,$$

belonging to the reciprocal type [2, 3; 2, 1 : 0], and the Σ is in fact the discriminant of one binary quadratic multiplied by the connective between it and another.

So if we take the graph represented in (a), Fig. 43,

$$\Sigma\,(O_1-O_2)(O_1-H)(O_2-K)$$

will represent an invariant to the type [2, 2 ; 1, 1 ; 1, 1 : 0]. If, however, we were to substitute H_1, H_2 in lieu of H and K, so as to form the hydroxyl graph of Fig. 43, (b), it would not be true that $\Sigma\,(O_1-O_2)(O_1-H_1)(O_2-H_2)$ would represent an invariant to the type [2, 2 ; 2, 1 : 0] ; on the contrary it would be zero. But hydroxyl is *not an invariant*, for to the combination of a quadratic and a linear form there appertains no invariant of the second degree in the coëfficients of each of them. This may be easily proved by the rule I have given at the commencement of this paper. I have gone through this calculation for the benefit of those new to the subject and to show how

the arithmetical "rule of multiplicity" is to be applied. Had I been writing solely for algebraists it would have been unnecessary to prove so familiar a fact. We have here $i=2$, $j=2$, $i'=1$, $j'=2$, $w=\dfrac{ij+i'j'}{2}=3$. To find $(w:\ i,j;\ i',j')$ we have to count the combinations

$$\begin{array}{c|c} 2.1 & 0.0 \\ 2.0 & 0.1 \\ 1.1 & 0.1 \\ 1.0 & 1.1\,; \end{array}$$

the number of these is 4. Again to find $(w-1:\ i,j;\ i';\ j')$ we have to count the combinations

$$\begin{array}{c|c} 2.0 & 0.0 \\ 1.1 & 0.0 \\ 1.0 & 0.1 \\ 0.0 & 1.1\,, \end{array}$$

of which the number is also 4. Hence

$$\Delta\,(3:\ 2,2;\ 1,2)=4-4=0.$$

So that hydroxyl, being of the type [3: 2, 2; 1, 2], cannot be an invariant.

So far then the supposed law is safe; but I think I see other difficulties in the way of its application to heteronymous types, so that if it shall be capable of being made universally applicable, other parts of the graphical theory, as it has been laid down, will possibly require reconsideration. What I advance is to be regarded not as dogmatic but as tentative and open to correction.

It is obvious that not every chemico-graph, potential or even actual, corresponds to an invariantive derivative. Of this I have already given examples. Were the case otherwise we should have surprised the secret of nature, for, as we know how to obtain all possible fundamental forms to binary quantics, we should know *à priori* all possible compound radicals. As a matter of fact the cases of algebraical invariance in nature seem to be rare and rather the exception than the rule. Thus while muriatic acid, $(H-Cl)$, is an invariant, self-saturating hydrogen, $(H-H)$, is a non-invariant, there being a linear invariant to two linear forms but not to a single one. In like manner ozone (Fig. 16) is also non-invariantive, there being no cubic invariant to a quadratic form. But there is an essential difference to be observed between the two cases. A graph consisting of a single or an odd number of

bonds between two atoms of the same kind can *never*, for any species of such atoms, be invariantive, because no covariant of the second order in the coëfficients can have an *odd* weight. If that were possible, then, by the theorem of reciprocity, a quadratic function could have an invariant or covariant of an odd weight, which is, of course, not true. Whereas a triangle of n-ads, although not picturing an invariant when $n=2$, does do so when $n=3$ or any higher number. When a homonymous graph is given in weight (the number of bonds) and in order (the number of atoms) two of the elements of its type $(w: i, j)$ say w, j are known and the third i is left indeterminate. For all values of i which make $\Delta\ (w: i, j)$ greater than zero, there will be one or a pluralty of such graphs according to the value of Δ. If no value of i makes Δ greater than zero, there will be no such graph possible, but it is not necessary to ascertain this to make an indefinite number of trials, for it is obvious that for all values of i equal to or greater than w, Δ has the same value viz. $\Delta\ (w: \infty, j)$, since the condition that a number w shall not be made up of nmbers greater than i, when i is equal to w, becomes nugatory.

It will be instructive to consider the case of $w=5$, $j=3$, and consequently the free valence $\varepsilon = 3i - 10$; this implies that i must be at least equal to 4. But if we take $i=4$, $\varepsilon=2$, as there is no covariant to a binary quartic whose order is 3 and degree 2, we may be sure that $\Delta\,(5: 4, 2) = 0$. Hence we have only to consider the case of $i = w = 5$, $\varepsilon = 5$. $\Delta\,(5: 5, 3)$ is the number of covariants of the fifth order and fifth degree to a cubic of which there is but *one*, formed by the multiplication together of the Hessian and skew-covariant. If now we proceed to form the graph corresponding to the type $[5: 5, 3]$, we have the choice of two figures, 17, 18. In the former figure there are three degrees of vacancy from saturation at A and one at each of the points B, C. In the latter, one at A and two at each of the points B and C. The graph, we must recollect, is to correspond to a cubic covariant of the fifth degree to a fifthic which is unique and indecomposable. This enables us to fix upon the true representation. It cannot be the graph of Fig. 17, for that may be considered as generated by the combination of one isolated nitrogen atom with two atoms of nitrogen, B, C, connected by five bonds; two of these being subsequently welded together and bent out into the angle having A at its vertex. [The hypothetical nitrogen pair exists in chemistry but not as an algebraical invariant.] Hence the true figure can but be that given in Fig. 18, where the free valence is separated into the parcels 2, 1, 2, and

not as in Fig. 17 into the parcels 1, 3, 1. And it should be observed that, for all higher values of i beyond 5, this will continue to be the one and only true graph to the corresponding covariant. It thus appears that every given homogeneous graph has an intrinsic character of capability or incapability of respondence to algebraical in- or co-variance, irrespective of the particular valence assigned to its atoms, and it is natural to suppose that there must be some immediate intrinsic criterion for determining this character, so as to dispense with the necessity of any algebraical considerations to establish it; but if such criterion exists, I have not yet been able to make out what it is.* In common with this view we may consider the theory of reciprocity of algebraical derived forms. It has already been stated that to every m-ad of n-ad atoms having a given number of bonds corresponds an n-ad of m-ad atoms with the same number of bonds. As for example, to a quasi carbon-ad (so to say) of sulphur will correspond a quasi sulphur-ad of carbon, the number of bonds and consequently the amount of free atomicity remaining the same in the two molecules. This suggests the possibility of there being some mode of passing from a graph to its reciprocal (this reciprocity being seemingly of quite a different kind from that which connects correlated girders or frameworks in graphical statics). I offer the subjoined instance of such transformation tentatively and with a view to stimulate inquiry, rather than as possessing any assurance of the validity of the process employed.

Suppose the case of $i = 4$, $j = 2$, $w = 4$; the one and only corresponding graph will be a system of 4 bonds connecting two atoms A, B. If now we take a pair of these bonds, stretch them out, weld them together and form a knot between them at C, and in like manner convert the other pair of bonds into a pair knotted at D, we shall have a graph consisting of a simple quadrilateral which will correspond to the case of $i = 2$, $j = 4$.

Again, suppose $i = 6$, $j = 4$, $w = 12$. We may consider either of the graphs quasi in Figures 19, 20. In the first of these figures we may take four bonds connecting respectively AC, CA, AD, DB, stretch and weld them together and form a knot between them at a new point E which will then be attached by four bonds to the atom $ABCD$. I mean that we may stretch out AC, CB, to meet in E (Fig. 21) and have EC common, and in like manner stretch out AD, DB to E and have ED common and then knot together the

* The law of reciprocity, however, exemplified in p. 74 can obviously be made to supply the criterion in question.

four bonds of the strings at E. In like manner we may form another knot F with bonds through AB, BC, AD, DC, and shall thus obtain the reciprocal graph of Fig. 21, where now $i = 4, j = 6, w = 12$. So again it will be found that we may distort Fig. 20 (if I can trust to my recollection of the result of previous work) in two different ways into a reciprocal graph.

At the risk of provoking the ire or ridicule of my chemical friends and the chemical public, I will venture to throw out a few remarks on the substructure, so to say, of the accepted theory of atomicity and to offer a suggestion as to a possible mode of getting rid of some imperfections under which it appears at present to labor. First there is the inconsistency of admitting the isolated existence of single atoms of mercury, cadmium and zinc, as monads with their bonds or tails absorbed or suppressed or else swinging loose and unsatisfied in direct opposition (as it seems to me) to the fundamental postulate of the theory. Next, one cannot get over a somewhat uncomfortable feeling at the representation of isolated oxygen in the state of ozone by a triangular graph, which, although conceivable, is supported by no analogous case unless that of baric peroxide, or any similar graph, be regarded as such. Thirdly, there is the vague and unsatisfactory (not to say unthinkable) explanation of the variability of the valence of a given atom by what Dr. Frankland calls "the very simple and obvious assumption that one or more pairs of bonds belonging to the atom of an element can unite and having saturated each other become, as it were, latent."

Now these stumbling blocks to the acceptance of the theory may be removed by one simple, clear and unifying hypothesis, which will in no wise interfere with any actually existing chemical constructions. It is this: leaving undisturbed the univalent atoms, let every other n-valent atom be regarded as constituted of an n-ad of *trivalent* atomicules arranged along the apices of a polygon of n sides. Thus, sextivalent, quinquivalent and quadrivalent atoms in their state of maximum valence will be represented by Figures 22, 23, 24, where the letters denote *trivalent atomicules.* When the valence is reduced by two we need only conceive any one of the side loops doubled or a new loop as formed by the coalescence of a pair of free bonds or tails, and when in the Figures 22 and 23 the valence is reduced by 4, we may in like manner either suppose existing loops doubled, or fresh ones inserted, or both changes to go on simultaneously, by the coalescence of two pairs of tails. We have thus a conceivable and conformable-to-analogy method of accounting for the varia-

bility in question. So likewise, a trivalent atom with maximum state of valence will be represented by Fig. 25, and when univalent by Fig. 26. Again, an isolated zinc element will have for its graph Fig. 1, (*b*), the two letters Z signifying the zinc atomicules, and so in like manner isolated cadmium and mercury may be represented. On the other hand O_2, isolated oxygen in its ordinary state, will be represented by the graph of Fig. 27, whilst ozone will have for its representative graph the well known Kekuléan hexad (which, in its importance to chemistry, would seem to vie with Pascal's mystic hexagons to geometry) represented in Fig. 28, where as in Fig. 27, each letter O represents an atomicule of oxygen. So an isolated element of carbon would be represented by the graph of Fig. 29.

This hypothesis of atomicules, if unobjectionable on other grounds, would not be open to the charge of having any tendency to disturb or complicate the existing graphology; for we should still be at perfect liberty to substitute for the graphs (*a*) of Figures 30, 31, 32 the abridged notation (*b*), and should naturally do so when considering the relations of atoms to each other. The beautiful theory of atomicity has its home in the attractive but somewhat misty border land lying between fancy and reality and cannot, I think, suffer from any not absolutely irrational guess which may assist the chemical enquirer to rise to a higher level of contemplation of the possibilities of his subject. I have therefore ventured to make the above suggestion.

Chemical graphs, at all events, for the present are to be regarded as mere translations into geometrical forms of trains of priorities and sequences having their proper ***habitat*** in the sphere of order and existing quite outside the world of space. Were it otherwise, we might indulge in some speculations as to the directions of the lines of emission or influence or radiation or whatever else the bonds might then be supposed to represent as dependent on the manner of the atoms entering into combination to form chemical substances. Such not being the case, what follows is to be considered as having relation to mere ***algebraical*** atoms, or atomicules (quantics) and their bonds which may be regarded as represented by the linear factors of such quantics.

Let us consider a symmetrical trivalent atomicule whose three bonds or rays make angles of 120° with each other. Calling τ, τ', τ'', the tangents of the angles which the axis of x makes with its rays, we have

$$\tau' = \frac{\tau + \sqrt{3}}{1 - \sqrt{3}\,\tau} \qquad \tau'' = \frac{\tau - \sqrt{3}}{1 + \sqrt{3}\,\tau},$$

so that its equation will be easily found to be

$$(1 - 3\tau^2)\,x^3 + (9\tau - 3\tau^3)\,x^2y + (9\tau^2 - 3)\,xy^2 + (\tau^3 - 3\tau)\,y^3 = 0\,,$$

which may be identified with the standard form

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0$$

by writing $\quad a = 1 - 3\tau^2 = -c, \qquad b = 3\tau - \tau^3 = -d.$

Suppose the three atomicules to become condensed into a single atom after the manner of the graph of Fig. 25. The combination will be represented by the cubic covariant (see Tables des Invariants et Covariants, Table V, annexed to Faà de Bruno's "Théorie des Formes Binaires")

$$(a_2 d - 3abc + 2b^3)\,x^3 + (3abd - bac^2 - 3b^2c)\,x^2y$$
$$+ (3bc^3 + 6b^2d - 3acd)\,xy^2 + (3bcd - ad^2 + 2c^3)\,y^3\,,$$

which, for the present case, becomes

$$2\,(1 + \tau^2)^3[(3\tau - \tau^3)\,x^3 + (9\tau^2 - 3)\,x^2y + (3\tau^3 - 9\tau)\,xy^2 + (1 - 3\tau^2)\,y^3].$$

Hence the new ray-directions will have for their equation

$$-\,dx^3 + 3cx^2y - 3bxy^2 + ay^3 = 0,$$

or the pencil of the atom will be identical with that of each of the separate atomicules, but accompanied with a rotation (whatever that may mean) of the whole pencil of rays through a right angle in its own plane. Again, suppose that only two atomicules are brought into connexion as in (a) of Fig. 30. The quadricovariant which expresses the atom (Faà de Bruno *ante*) is

$$(ac - b^2)\,x^2 + (ad - bc)\,xy + (bd - c^2)$$

which here becomes $\qquad -\,(1 + \tau^2)^3\,(x^2 + y^2).$

Hence the ray-directions will be given by the equation

$$y^2 + x^2 = 0, \qquad y = \pm\, x\sqrt{-1}\,,$$

which we may, if we please, according to the usual convention concerning the square root of minus unity, explain by supposing that the original rays are situated in planes perpendicular to the joining line XX, and that these are replaced by two rays lying in opposite directions along the line XX, where the atomicules are condensed into one atom. But it would be idle to pursue this speculation further.

The most remarkable point in the theory which I have endeavored to unfold in the preceding pages is the relation between it and that of reciprocal types.

We have seen that the graph to an invariant of one type read off as it stands (each bond being construed as the sign *minus*) with the sign Σ prefixed expresses an invariant of the reciprocal type.

This rule may be extended from homogeneous to heterogeneous graphs, provided only that the reciprocity be *total*, by which I mean that every i and every j in the type $[i, j;\ i', j';\ i'', j'' \ldots\ldots : 0]$ are interchanged. It may be observed, in passing, that in the case of types to which resultants belong, the type is identical in form with its total reciprocal. As ex. gr. boric anhydride (consisting of two of boron and three of oxygen) is of the type [3, 2; 2, 3 : 0].

On referring to "System of Cubic and Quadratic," Salmon's Lessons, third edition, p. 179, it will be seen that besides the resultant there is another invariant represented in Dr. Salmon's notation by "Δ (0, 2) $\times$ I (2, 1)"; a linear combination of these two with arbitrary multipliers will express the most general form belonging to the type in question.

From the property of these types being their own complete reciprocals, it follows that a complete set of independent graphs of any such type will represent the constitution of a complete set of independent forms belonging to the type. Thus, in the case suggested by boric anhydride we have the two independent graphs of Figures 33, 34. Hence the complete representation of the invariants appertaining to the self-reciprocal diadelphic type [3, 2; 2, 3: 0] is $\lambda U + \mu V$, where U is the resultant $(a - \alpha)(a - \beta)(a - \gamma)(b - \alpha)(b - \beta)(b - \gamma)$ and V is $\Sigma\,(a - \gamma)(a - \beta)(b - \alpha)(b - \gamma)(b - a)(\beta - \alpha)$. U is derived from the graph of Fig. 33 by replacing the several O's by α, β, γ, and the B's by a, b, and V in like manner from the graph of Fig. 34.* This latter graph is replaceable by the disjoined graph of Fig. 35, to which, by the rule for combination of graphs, it is easily seen to be equivalent.

Hence, instead of $\lambda U + \mu V$ we may write $\lambda V + \mu V'$ where $V' = \Sigma\,(a - \beta)^2 (a - b)^2 (a - \gamma)(b - \gamma)$; a, b of course will be understood to be the roots of a general quadric and α, β, γ of a general cubic. A very good similar instance of this kind of equivalence is afforded by the quadrinvariant of a quartic whose type is [4, 2 : 0]. The reciprocal of this, viz. [2, 4 : 0], may be represented, either by the connected graph of Fig. 36, or by the disjoined one of Fig. 37, and accordingly the noted quadrinvariant $ae - 4bd + 3c^2$ may be expressed (to a numerical factor près) either by the sym-

* In this figure on the side opposite to BB, a third letter O has been accidently omitted.

metrical function $\Sigma\,(a-c)(a-d)(b-c)(b-d)$ corresponding to the first, or by $\Sigma\,(a-b)^2(c-d)^2$ corresponding to the second graph. Again, let us consider the contrary types, [4, 3: 0], [3, 4: 0]. The former has for its graph Fig. 38, and admits of no other representation. This gives

$$\Sigma\,(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2$$

for the discriminant of the cubic which belongs to the contrary type. The latter may be figured chemically by the graph (consisting of two molecules of boron) of Fig. 39, or by the equivalent Fig. 27 (capable of being derived from it by the mechanical rule for conversion of graphs). These two latter, algebraically speaking, will be pseudographs, because $\Sigma\,(\alpha-\beta)^3(\gamma-\delta)^3$ and $\Sigma\,(\alpha-\beta)(\beta-\gamma)(\gamma-\delta)(\delta-\alpha)(\alpha-\gamma)(\beta-\delta)$ are each zero. The graph of Fig. 27 may be mechanically converted, in the manner shown in the preceding case, into the graph of Fig. 40; but the type of the colligation remains unaltered by this conversion and whichever of the two we employ, we obtain

$$\Sigma\,(\alpha-\beta)^2(\gamma-\delta)^2(\alpha-\gamma)(\beta-\delta)$$

as the representation in terms of the roots, of the cubic invariant to the quartic, viz. to a numerical factor près

$$ace-b^2e-ad^2+2bcd-c^3\,.$$

Thus we see that the graphical method suggested by the theory of atomicity is a real instrument not merely for the representation but also for the calculation and comparison of algebraical results. The important bearing upon it of the principle of contrary or reciprocal graphs, renders it desirable that I should put the algebraical theory or law of reciprocity, in its most complete form, before my readers; it will form the subject of Appendix 2.

I might have noticed explicitly at the commencement of this paper, instead of tacitly assuming it as I have done, that the chemical fact of a compound molecule playing the part of an atom with a valence equal to the free valence of the radical, is the precise homologue to the algebraical fact that every invariant or covariant of a covariant, or set of covariants, to a quantic, or system of quantics, is itself an invariant or covariant to such quantic, or system of quantics; and again that Regnault's chemical principle of substitution and the algebraical one of emanation* are identical; and again, the

* By which I mean in this place the operation upon an invariant or covariant of the symbol $(a'\delta_a+b'\delta_b+\ldots)$ performed any number of times in succession; a, b, for instance, may refer to Hydrogen $(ax+by)$ and a', b' to Chlorine $(a'x+b'y)$, and then the emanantive operator, according to a notation used, if I mistake not, by Professor Clerk Maxwell in his theory of poles, might be denoted by $Cl\,\delta_H$.

Plate I.

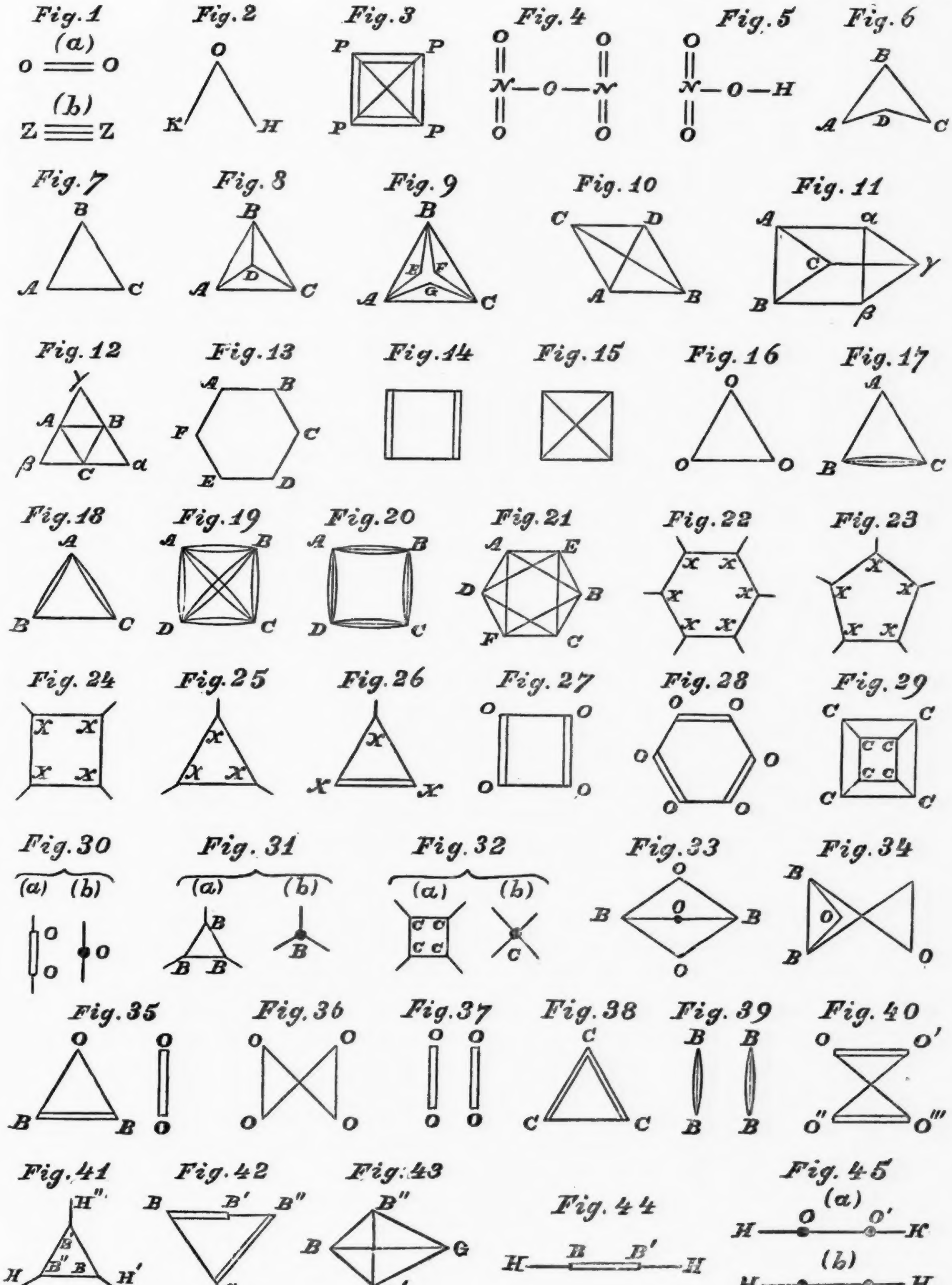
Fig. 1
Fig. 2
Fig. 3
Fig. 4
Fig. 5
Fig. 6
Fig. 7
Fig. 8
Fig. 9
Fig. 10
Fig. 11
Fig. 12
Fig. 13
Fig. 14
Fig. 15
Fig. 16
Fig. 17
Fig. 18
Fig. 19
Fig. 20
Fig. 21
Fig. 22
Fig. 23
Fig. 24
Fig. 25
Fig. 26
Fig. 27
Fig. 28
Fig. 29
Fig. 30
Fig. 31
Fig. 32
Fig. 33
Fig. 34
Fig. 35
Fig. 36
Fig. 37
Fig. 38
Fig. 39
Fig. 40
Fig. 41
Fig. 42
Fig. 43
Fig. 44
Fig. 45

modern notion of two semi-molecules, simple or compound, combining or uniting to form a chemical substance is tantamount to the construction of an invariant, the connective (or in Professor Gordan's language, the final "Ueberschiebung") of a quantic, or of the derivee of a quantic or a set of quantics, with itself. So again, it will hereafter be seen* that Hermite's law of reciprocity applied to quantic systems and stated in its widest terms, amounts to affirming in chemical language that in any compound an arbitrarily selected group of m n-adic atoms may be replaced by a group of n m-adic atoms, but how far this law of replacement has objective validity in the chemical sphere, I am not able to say.

Attention might also have been called to the fact that every chemicograph may, for anything that has been shown to the contrary, and probably in all cases does admit of algebraical interpretation, provided that each given atom however often repeated in a graph counts as a distinct quantic with its own distinct set of coëfficients. I do not know whether chemists are of opinion that every chemico-graph exists or is capable of existence in nature; if this is not the case, the condition of the possibility of such existence (should it be discovered) must admit of being stated in mathematical terms. The condition for its existence in algebra may be gathered from what precedes, to be certainly for monadelphic types and probably in all cases, as follows, viz: *if the difference between every two letters of an algebraically existent graph be raised to the power whose index is the number of bonds connecting them, the permutation sum of the product of those powers must not vanish.* Finally, an irreducible covariant is the homologue of a compound radical. Thus we see that chemistry is the counterpart of a province of algebra as probably the whole universe of fact is, or must be, of the universe of thought.

APPENDIX 1.

REMARKS ON DIFFERENTIANTS EXPRESSED IN TERMS OF THE DIFFERENCES OF THE ROOTS OF THEIR PARENT QUANTICS.

SINCE the preceding matter was written, in dwelling upon the law of reciprocal graphs, I came to what appeared to be a formidable difficulty in

* In Note D to Appendix 2. The proposition stated in the text results from the joint effect of the law of substitution or emanation combined with Hermite's law extended to quantic systems.

the way of its reception, a very lion in my path, so formidable that, for a time, I thought that it would be necessary, either to abandon this law, or else to admit the unwelcome conclusion that not every type of invariant was susceptible of graphical representation.

But further consideration has shown me that this apprehension was entirely groundless owing to an algebraical fact on which I had not previously reflected, but which this difficulty forced upon my notice. The difficulty in question arose out of the expressions given by M. Hermite and le père Joubert respectively for the skew invariants of the binary quintic and sextic. I shall first address myself to the consideration of the former. Following Dr. Salmon's notation (Lessons, 3d Edition, p. 230), let $\alpha, \beta, \gamma, \delta, \varepsilon$ be the roots of a quintic, and let

$$F = (\alpha-\beta)(\alpha-\varepsilon)(\delta-\gamma) + (\alpha-\gamma)(\alpha-\delta)(\beta-\varepsilon)$$
$$G = (\alpha-\beta)(\alpha-\gamma)(\varepsilon-\delta) + (\alpha-\delta)(\alpha-\varepsilon)(\beta-\gamma)$$
$$H = (\alpha-\beta)(\alpha-\delta)(\varepsilon-\gamma) + (\alpha-\gamma)(\alpha-\varepsilon)(\delta-\beta)\,.$$

Then it will be found as will presently be shown that the product $F.G.H$ is a symmetrical function of the four roots β, γ, δ, ε, consequently, on forming four other similar products symmetrical in respect to $\alpha, \gamma, \delta, \varepsilon : \alpha, \beta, \delta, \varepsilon : \alpha, \beta, \gamma, \varepsilon : \alpha, \beta, \gamma, \delta$ respectively, the product of these five products will be symmetrical in respect to $\alpha, \beta, \gamma, \delta, \varepsilon$ and being a function of the differences of the roots of order 18 and of weight 45, *i. e.* of the type [45 : 5, 18], must be (paying no attention to a mere numerical factor) I, the skew invariant to the quintic.

Now consider the type reciprocal to this, [45 : 18, 5], (monadelphic like the preceding), and expressing the invariant of the fifth order to an octodecadic. Suppose this has a graph. It will follow from the law of reciprocal graphs that I may be expressed under the form

$$\Sigma\,(\alpha-\beta)^a(\alpha-\gamma)^b(\alpha-\delta)^c(\alpha-\varepsilon)^d(\beta-\gamma)^e(\beta-\delta)^f(\beta-\varepsilon)^g(\gamma-\delta)^h(\gamma-\varepsilon)^k(\delta-\varepsilon)^l,$$

where $a+b+c+\ldots\ldots = 45$ and each letter $\alpha, \beta, \gamma, \delta, \varepsilon$ is conditioned to appear the same number of times, which at first might seem contradictory to what has just been established, but in reality is in perfect accordance with it. For imagine the product of the 15 quantities $FGHF'G'H'F''G''H''F'''G'''H'''F^{\text{iv}}G^{\text{iv}}H^{\text{iv}}$ to be actually written out giving rise to 2^{15}, or 32768 terms, and to each of these terms prefix the sign Σ indicating that the sum is to be taken of the 120 values which it assumes on permuting the 5 letters $\alpha, \beta, \gamma, \delta, \varepsilon$. The sum of

all these partial sums is $120I$; hence some, at least, of them cannot vanish. Let ΣT be any one that does not vanish. Then ΣT is a function of the differences of the roots of the same weight and order as the entire expression; it is therefore to a numerical factor près identical with I, just as every fragment of a mirror is itself a mirror, or as every particle of diamond dust, a diamond.

Thus, as many distinct non-vanishing forms as there may be of ΣT, so many different graphs to the quint-invariant of a binary octodecadic shall we be able to construct agreeing respectively with the different representations of I of the form

$$\Sigma\,(\alpha-\beta)^a(\alpha-\gamma)^b(\alpha-\delta)^c.\;.\;.\;.\;.\;.$$

and it is probable that the virtual equivalence of all these several graphs may admit of being made out by inspection, as we saw was the case with the two graphs (one dissociated, the other connected) corresponding to the two algebraical representatives of the quadrinvariant of a quartic. Thus, what seemed, at first sight, to be fatal to the admissibility of the algebraico-graphical theory only serves to set in a clearer light its value as an instrument of research.

If we analyse M. Hermite's form of the skew invariant* to the quintic we shall see that it depends upon this simple but not obvious fact, that writing

$$F=(c,\,d)\,(a-b)+(a,\,b)\,(c-d)$$
$$G=(b,\,d)\,(a-c)+(a,\,c)\,(d-b)$$
$$H=(b,\,c)\,(a-d)+(a,\,d)\,(b-c)$$

and interpreting any such quantity as $(a,\,b)$ to mean either 1 or $(a+b)$ or ab the product FGH is a symmetrical function of a, b, c, d, because on interchanging any two letters (say ex. gr. c, d) that one of the three quantities F, G, H (in this example H) in which those two letters are affected with the same sign, will remain unaltered in value whilst the other two (here G and F) change, each into the negative of the other.

Consequently we may interpret $(a,\,b)$ to mean $(e-a)\,(e-b)$ and then the product of the five products corresponding to FGH is a function of the

*I am wont to compare in my mind this symmetrical and translucent form to the Pitt Diamond and Père Joubert's to the Koh-i-Noor. In Note D to Appendix 2 a method is given whereby these forms may be transmuted into one another subject, however, to the bare possibility that the one, put into the algebraical alembic at a certain stage of the process, instead of passing into the other may, so to say, evaporate and be reduced to nothing. In the theory of forms, all-embracing Zero is the source and reconciler of contradictions, because, algebraically speaking, *everything is contained in nothing*, and so in a morphological sense "nought is everything" though not "everything is nought."

coëfficients which expressed in terms of the differences of the roots will be of the weight 15 and of the order $1\cdot 6 + 4\cdot 3$ or 18 because in one of the 5 products each letter will enter in six dimensions and in each of the other 4 products in 3 dimensions; thus in FGH, e^6 will appear, but in each of the other 4 products e^3 will be the highest power of e. Hence the quindenary product is the invariant in question. No further step is necessary, the proof is complete as stated.

This remark will enable us to illustrate the process of transformation, which I have compared with grinding a diamond into dust, by an example that can be completely pursued to the end. For let us now regard a, b, c, d as the roots of a binary quartic; then $(\overline{a - b} + \overline{c - d})(\overline{a - c} + \overline{d - b})(\overline{a - d} + \overline{b - c})$ will be a differentiant thereto of weight 3 and order 3; it will, in fact, represent the root-differentiant of the skew sextic covariant.

Imagine this multiplied out without disturbing the marks of coupling so as to give 8 terms or fragments analogous to the 32768 fragments spoken of in the preceding case. These terms will be of only four different patterns, one of the pattern $(a - b)(a - c)(a - d)$, three of the pattern $(a - b)a - c)(b - c)$, three of the pattern $(a - b)(b - c)(d - b)$ and one of the pattern $(c - d)(d - b)$ $(b - c)$. Prefixing Σ to each of these pattern terms to signify the sum resulting from the 24 permutations of a, b, c, d, we know *à priori* that not all of these can be zero since a linear function of them will be 24 times the differentiant in question, and on examination we find that the second and fourth Σ will vanish, but that the first and third will not. Accordingly, we shall have two new expressions $\Sigma\,(a - b)(a - c)(b - c)$, $\Sigma\,(a - b)(b - c)(b - d)$, each of which represents a differentiant of the same type as the original one, and this type being monadelphic or enparametric, the original product and these two sums will only be different representations of the same differentiant. Thus we see that each independent form belonging to a given type is susceptible (when expressed as a function of the differences of the roots) of a number of distinct phases, or, as we may express it, an algebraical form, in this theory, is in general polyphasic and accordingly its Icon or linkage exponent will be in general polygraphic, and each phase will have its own appropriate graph. It is a work of some difficulty, in general, to recognize the substantial identity of the different phases of the same algebraical form, and in like manner it may not, in all cases, be easy to recognize the substantial identity of the different graphs of its Icon, but sufficient has been shown to

indicate the possibility and method of establishing such identity. The more I study Dr. Frankland's wonderfully beautiful little treatise the more deeply I become impressed with the harmony or homology (I might call it, rather than analogy) which exists between the chemical and algebraical theories. In travelling my eye up and down the illustrated pages of "the Notes," I feel as Aladdin might have done in walking in the garden where every tree was laden with precious stones, or as Caspar Hauser when first brought out of his dark cellar to contemplate the glittering heavens on a starry night. There is an untold treasure of hoarded algebraical wealth potentially contained in the results achieved by the patient and long continued labor of our unconscious and unsuspected chemical fellow-workers.

We have seen that M. Hermite's beautiful expressions for the skew invariant of the quintic proves its own character. A similar analysis may be applied to père Joubert's equally beautiful and even more remarkable expression for that of the sextic. M. de Bruno's statement of this, Table IV^{10}, contains two very perplexing typographical errors, viz. 4th line from foot of page, in V_0, $x_1x_2(x_\infty + x_0 - x_3 - x_2)$ should read $x_1x_2(x_\infty + x_0 - x_3 - x_4)$, and 3d line from foot of page, in W_0, $x_2x_4(x_2 + x_3 - x_\infty - x_0)$ should be $x_2x_4(x_1 + x_3 - x_\infty - x_0)$. Moreover, the form in which the expression is presented in M. de Bruno's pages tends to mask its true nature and to suggest an analogy, which has no existence in fact, between it and M. Hermite's form; the latter is intrinsically a quinary group of triadic products, but such representation in the case of M. Joubert's form is purely conventional and confusing, it really being a single indecomposable quindenary product. Call a, b, c, d, e, f the six roots of a sextic, and let ab; cd; ef be any one of the 15 *duadic synthemes** which can be formed with them, and

$$F = \pm \left\{ \begin{array}{r} ab.\,\overline{c+d-e-f} \\ +\, cd.\,\overline{e+f-a-b} \\ +\, ef.\,\overline{a+b-c-d} \end{array} \right\}$$

The external sign is arbitrary, but must be considered as *determined* once for all for each of the 15 values of F. The product of these 15 values is a symmetrical function of the roots. For suppose any two letters, as a, b, to be

* A duadic syntheme of $2n$ letters is a combination of n duads containing between them all the letters. In it the order of the duads and of the letters in each duad is disregarded. Hence the number of such is $\frac{\Pi 2n}{2^n \Pi n}$ or $1\cdot3\cdot5\cdots(2n-1)$. For an odd number of letters simple synthemes do not exist but in lieu of them we may construct diplo-synthemes containing every letter taken twice over.

interchanged; then three of the factors F in which a and b are coupled will undergo no change, but the remaining twelve will evidently be resoluble into six pairs reciprocally related, so that each F of a pair is transformed either into the other or into its negative and on either supposition the product of the pair remains unaltered in value. Also this product is a differentiant, for $\Sigma\delta_a$ operating on any one factor evidently reduces it to zero. It is also of the weight 45 and of the order 15. Hence the product of the fifteen values of F is the skew invariant to the sextic.

It seems desirable to make the ***differentiantive*** character of the form self-apparent. This may be done by virtue of the remark that $\pm F$ may be replaced by the form

$$\left\{\begin{array}{l} \quad (a-d)\ (b-f)\ (c-e) + (a-f)\ (b-d)\ (c-e) \\ + (a-c)\ (b-e)\ (d-f) + (a-e)\ (b-c)\ (d-f) \\ + (a-c)\ (b-f)\ (d-e) + (a-f)\ (b-c)\ (d-e) \\ + (a-d)\ (b-e)\ (c-f) + (a-e)\ (d-b)\ (c-f) \end{array}\right\}$$

This sum contains 64 terms, of which 48 are the terms in F taken 4 times over, and the other 16 are the 8 quantities ace, bdf, acf, bde, bce, adf, bcf, ade, each appearing twice with opposite signs. If we expand the product of the 15 values of F, we shall obtain 35,184392,568832, or upwards of 35 billions of terms distributable among a certain number of patterns; on prefixing Σ to one of each pattern a certain number of such sums will be zero, but the remaining ones of which there must be some (and there will probably be a very large number) will all be (except as to a numerical multiplier) identical with each other and with père Joubert's formula. We see by these examples that there is a sort of polymorphism or pheno-polymorphism, as it may be termed, which is of a much more superficial character than and ought to be carefully distinguished from true polymorphism, eteo-polymorphism as we may call it, and this distinction as it has a marked bearing upon the theory of algebraical linkages, it is reasonable to expect may not be without importance in the study and construction of chemical graphs. Although I have been dealing, in what precedes, with particular cases, the reasoning is general in its nature and leads to conclusions which I will proceed to express in exact terms.

Let us understand by a permutation-sum of a function of letters belonging to one or more sets (n, n', n'', ... being the number of letters in the respective sets) the sum of the $\Pi n \Pi n' \Pi n''$ . . . . . values which the function assumes when the letters in each several set are permuted *inter se;* and let us under-

stand by a monomial differentiant one which (with the usual convention as to $a=1$) may be expressed as a permutation-sum of a single product of differences of roots of the parent quantic, or quantic system; then in the first place it has virtually been proved, in what precedes, and is undoubtedly true that every monadelphic differentiant is monomial, and it may easily be proved in like manner that a differentiant of multiplicity k may be represented by the sum of k monomial differentiants.

For greater simplicity let us confine ourselves to the case of monadelphic invariants and let us consider any two such belonging to reciprocal types; then the algebraical value of either one, in terms of the roots of its parent quantic or quantic system, will be represented by the permutation-sum of the product of the differences of every two letters in the other taken as many times as there are connecting bonds between them, such letters being for this purpose regarded as the roots in question. Hence also we may derive the rule previously given for determining whether or not any given graph, in which the number of bonds is equal to half the toti-valence, represents or not an algebraical invariant—the condition of its doing so being that the permutation-sum of the product of the differences between the connected letters (each bond giving one such difference) shall be other than zero. This rule will stand good whether the type of the graph be monadelphic or not.

A very simple instance occurs to me of the monomial law for monadelphic types. Let α, β, γ be the roots of a cubic. It will easily be found that the type $(4:3,4)$ to which

$$\left((\alpha-\beta)^2+(\alpha-\gamma)^2+(\beta-\gamma)^2\right)^2$$

belongs is monadelphic; prefix to it the sign of summation, which is merely equivalent to multiplying it by 6. It will not be a monomial permutation-sum as it stands, but it may be replaced by $2\Sigma\,(\alpha-\beta)^2(\alpha-\gamma)^2$ or $\Sigma(\alpha-\beta)^4$ each of which monomial sums is a half of

$$\left((\alpha-\beta)^2+(\alpha-\gamma)^2+(\beta-\gamma)^2\right)^2.$$

Postscript. Subsequently to the printing of the foregoing sheets I have seen in an editorial notice in the English Journal *Nature* (March 14, 1878) a statement of the claims of Dr. Frankland to be the discoverer and first promulgator of the law of atomicity, and I appear unconsciously to have done injustice to this great English chemist by attributing the discovery to Kekulè. I derived my impression on the subject from the popular belief and from the account of it given by Wurz in his "Histoire des doctrines chémiques." If the facts of the case are as set forth in *Nature* and admit of no qualifying statements, I am unable to understand how such a discovery as that of valence or atomicity, which furnishes the master-key to our knowledge of the transformations of matter and raises chemistry to the rank of a mathematical and predictive science (it was previously only arithmetical), can have escaped receiving the award of a Copley Medal from the society in whose

Transactions it appeared. I can hardly imagine that, if the first announcement and proof of universal gravitation or the circulation of the blood had been communicated to the world in a paper inserted in the Philosophical Transactions in these days, its author would have failed to receive for it the highest mark of recognition in the power of the Royal Society of London to bestow, and in my humble judgment the law of atomicity in its far-reaching importance and the labor, and mental acumen required for its discovery, stands fully on a level with either of these great landmarks in the history of natural science. It seems also from the same article in *Nature* that my distinguished friend, Professor Crum Brown, to whose personal teaching at Edinburgh I owe the very slight acquaintance with the subject I can lay claim to, was the first to use the admirable method of chemico-graphs.

The conception of hydro-carbon graphs as "trees with nodes, branches and terminals" and the indispensable notion of constructing them by starting from "an intrinsic central node or pair of nodes, so as to get rid of the otherwise unsurmountable difficulty of having to recognize equivalent forms appearing several times over in the same construction," are exclusively my own and were used by me in my communications with Professor Crum Brown on the subject and stated by me in a letter to Professor Cayley, who has adopted them as the basis of his own isomerical researches. In the account of this method given in German chemical journals I am informed that all reference (or at least all adequate reference) to my name as the author of it "fine by degrees and beautifully less," has at length entirely evaporated. M. Camille Jordan was led by quite a different order of considerations and with quite a different object in view to a discovery of the same centres before me, but I was not acquainted with this fact when I rediscovered them and made the application above mentioned. The idea of this application stands in the same relation to Professor Cayley's perfected use of it, as his idea of the use to be made of the equation $\Delta\,(w: i, j) =$ the number of linearly independent covariants of the type $[i, j: ij - 2w]$ stands to my completed method founded thereon, for obtaining the scale and connecting syzigies of the irreducible covariants to a quantic, laying me thereby under an obligation which I should take it in very ill part if any translator of my papers on the subject failed to acknowledge in unmistakable terms.

The hydro-carbon graphs, it may be noticed, belong to the limiting case of chemico-graphs; where no cyclical system of bonds connects any groups of atoms in a graph, it becomes an arborescence.

I have found it a profitable exercise of the imagination, from a philosophical point of view, to build up the conception of an *infinite* arborescence and to dwell on the relations of time and causality which such a concept embodies. An example of the good to be gained by these limitless mental constructions (new tracts and highways, so to say, opened out in the all-embracing "grand continuum" which we call space) is afforded by the valuable applications to the theory of local probability and the integral calculus in general made by Professor Crofton (my successor at Woolwich) of his new idea of an infinite reticulation (warp and woof), every finite portion of which contains an infinite number of meshes, being formed by the crossings of two sets of parallel lines all infinitely extended in both directions and those of the same set equidistant and infinitely near to each other. So the largest idea of an arborescence is that of an infinite number of nodes with an infinite number of branches proceeding from each of them.

APPENDIX 2.

NOTE ON M. HERMITE'S LAW OF RECIPROCITY.

I TAKE for granted that the treatise of M. Faà de Bruno represents this theory as it at present stands, in which case it seems to have made no advance since it was first promulgated by M. Hermite in his well known paper in the

Cambridge and Dublin Mathematical Journal, 1854. It will be seen, however, I think from what follows, that it admits of being presented in a somewhat simpler and more general form. It rests essentially on the proposition of reciprocity in the theory of partitions that $(w: i, j) = (w: j, i)$, from which it follows as an immediate consequence that the number of arbitrary constants in the general covariant (or invariant) whose type is $[w: i, j]$, is the same as that whose type is $[w: j, i]$ since that number will be $\Delta(w: i, j) = \Delta(w: j, i)$ for each. Let now $\phi\,(a, b, c, \ldots. l)$ be any differentiant of the order j in the coëfficients, and of the weight w to a binary quantic $F(x, y)$ of the degree i in the variables; then ϕ is the root of a single covariant whose order is j and degree in the variables $ij - 2w$. Let ϕ be expressed (as from the definition of a differentiant must necessarily be possible) as a function of the differences of the roots $\alpha_1, \alpha_2, \ldots. \alpha_i$ of F when y is made unity. For any difference $\alpha_p - \alpha_q$ substitute $\frac{d}{dx_p} \cdot \frac{d}{dy_q} - \frac{d}{dx_q} \cdot \frac{d}{dy_p}$, and let ϕ be converted into $\dot{\phi}$ by this substitution. Now operate with $\dot{\phi}$ upon the product of the j forms $G(x_1, y_1)$, $G(x_2, y_2), \ldots G(x_i, y_i)$, $G(x, y)$ signifying the general form of the degree j in the variables, and after the operation has been performed turning each subscript x into x and each subscript y into y, after the manner of Professor Cayley's original method of generating invariants or covariants as "Hyperdeterminants," we shall thus obtain an in- or co-variant to a form of the degree j which will be of the order i in the coëfficients and of the degree $ij - 2w$ in the variables, for there are w factors in $\dot{\phi}$ and each factor is of the second dimension in two of the x's and the corresponding two y's. Thus we shall have passed from a form of the type $[i, j: ij - 2w]$ to another of the type $[j, i: ij - 2w]$, or which is the same thing, from one of the type $[w: i, j]$ to another of the type $[w: j, i]$.

This latter may be called the *image* of the first. For facility of reference, let the number of arbitrary parameters in the one and the other type be called the multiplicity. If we repeat upon this image the process by which it was deduced from its primitive, we shall obviously get back the original type, but it by no means follows that if the multiplicity exceed unity, we shall get back the primitive form itself. It may be possible to revert to the same type without reverting to the same individual specimen of it;* and such, we shall presently see, is what in general happens.

*Just as, if I rightly understand the explanation given of fluorescence, a ray of light may give birth to some other form of motion and that again to another ray of light but of a different color from the first. The theory of reciprocity treated of in the text is, in fact, a theory of alternate generation.

Before proceeding further I shall give a very simple methodical rule for finding the image to any given invariantive form. Since, for any given value of i, the form and its image are each given when their root-differentiants are respectively given, it will be sufficient to assign the law for passing from the differentiant of the primitive to that of its image.

For this purpose, let the given in- or co-variant be expressed in terms of symmetrical functions of the roots of the quantic when the leading coëfficient (a), is made equal to unity. Then it will consist of terms, any one of which, apart from its numerical coëfficient, will be of the form

$$\Sigma\,(\alpha_1\alpha_2\ldots\ldots\alpha_\lambda)^0\,(\beta_1\beta_2\ldots\ldots\beta_\mu)^1\,(\gamma_1\gamma_2\ldots\ldots\gamma_\nu)^2\,(\delta_1\delta_2\ldots\ldots\delta_\pi)^3\ldots\ldots$$

$\alpha_1\alpha_2\ldots\alpha_\lambda$, $\beta_1\beta_2\ldots\beta_\mu$, $\gamma_1\gamma_2\ldots\gamma_\nu$, etc. being all distinct and comprising between them *all* the i roots and of course $\mu+2\nu+3\pi+$ etc. will be equal to the weight; to pass from a differentiant expressed in terms of roots of a given quantic to the expression in terms of coëfficients of the allied quantic of its image it will be found that the only thing necessary is to change any such factor as α^λ (where α is any root of the given quantic) into C_λ, the coëfficient of the term containing y^λ in the allied one. This rule is a consequence (obtainable by ordinary algebraical processes) from the method above explained, where it is to be borne in mind that in order to obtain the image from the given form we have only to substitute for each root α_κ which occurs in φ, the fraction $\dfrac{dx_\kappa}{dy_\kappa}$ and to multiply the result by such a power of $\dfrac{d}{dy_1}\cdot\dfrac{d}{dy_2}\ldots\ldots\dfrac{d}{dy_\kappa}$, as will just serve to make it integral. A much simpler demonstration of this rule will be given in the sequel, and it will be shown that it not only holds good for deriving the leading term of the reciprocal (in the case of a covariant) from that of the primitive (i. e. the root-differentiant of the one from the root-differentiant of the other) but that it is applicable to deriving the whole of one expression from the whole of the other.

As an example, take the differentiant whose type is [3: 3, 3], the root or base of the skew covariant to a cubic $(a, b, c, d \between x, y)^3$. Its value is $a^2d-3abc+2b^3$; expressed in terms of the roots α, β, γ, making $a=1$, this becomes

$$\alpha\beta\gamma-\frac{3\,(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)}{9}+2\,\frac{(\alpha+\beta+\gamma)^3}{27},$$

$$\text{or }\frac{1}{27}\left\{27\alpha\beta\gamma-9\,(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)+2\,(\alpha+\beta+\gamma)^3\right\},$$

$$\text{or }\frac{1}{27}\left\{2\Sigma\alpha^3-3\Sigma\alpha^1\beta^2+12\alpha\beta\gamma\right\},\ i.\,e.\ \frac{1}{27}\left\{2\Sigma\alpha^0\beta^0\gamma^3-3\Sigma\alpha^0\beta^1\gamma^2+12\alpha^1\beta^1\gamma^1\right\}.$$

Applying the rule, this becomes converted into

$$\frac{1}{27}\left\{6C_0^2C_3 - 18C_0C_1C_2 + 12C_1^3\right\},$$

or, reverting to the letters a, b, c, d, the image becomes the primitive affected with the factor $\frac{6}{27}$ and may be seen to be its own conjugate. Or again, let the primitive be the discriminant of a cubic, *i. e.*

$$\frac{1}{27}(\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2 \text{ or } (\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha-\alpha\beta^2-\beta\gamma^2-\gamma\alpha^2)^2;$$

this is equal to

$$\Sigma(\alpha^2\beta^4 + 2\Sigma\alpha\beta^2\gamma^3 - 2\Sigma\alpha^3\beta^3 - 6\alpha^2\beta^2\gamma^2 - 2\Sigma\alpha\beta\gamma^4).$$

Hence, by our rule, the image will be

$$\frac{1}{27}(6c_0c_2c_4 + 12c_1c_2c_3 - 6c_0c_3^2 - 6c_2^3 - 6c_1^2c_4),$$

or, using a, b, c, d, e in lieu of c_0, c_1, c_2, c_3, c_4, we obtain the form

$$-\frac{6}{27}(abe + 2bcd - ad^2 - c^3 - b^2e),$$

i. e. $-\frac{2D}{9}$, where D is the well known quadrinvariant to a quartic $\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$.

Treating this quadrinvariant as a function of the roots of a biquadratic form and proceeding as before to form its image, we shall obtain a second image which will be a numerical multiplier of the original invariant.

But now let us consider the case of polyadelphic forms belonging to reciprocal types and for greater brevity, as the calculations are necessarily long, take a quantic of the self-contrary type $[w: i, i]$, as ex. gr. $[6: 4, 4]$ which belongs to the covariant of the fourth order and fourth degree to a quartic. This will be diadelphic; its general form is a linear combination of two products, one of the quartic itself by its cubinvariant, the other of the Hessian by the quadrinvariant. It will therefore have for its leading coëfficient the differentiant

$$\lambda a(ace + 2bcd - ad^2 - c^3 - b^2e) + \mu(ac - b^2)(ae - 4bd + 3c^2),$$

say $\lambda U + \mu V$. Let us first find the image of U. Expressed in terms of the roots $\alpha, \beta, \gamma, \delta$, it is

$$\frac{1}{6}(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)(\alpha\beta\gamma\delta)$$

$$+\frac{1}{48}(\alpha+\beta+\gamma+\delta)(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)$$

$$-\frac{1}{16}(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)^2-\frac{1}{16}(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)^3$$

$$-\frac{1}{16}(\alpha+\beta+\gamma+\delta)^2\alpha\beta\gamma\delta\,,$$

which is

$$\frac{[6](\alpha\beta\gamma^2\delta^2)}{6}+\frac{[24](\alpha\beta^2\gamma^3)+[48](\alpha\beta\gamma^2\delta^2)+[12](\alpha^2\beta^2\gamma^2)+[12](\alpha\beta\gamma\delta^3)}{48}$$

$$-\frac{[4](\alpha^2\beta^2\gamma^2)+[12](\alpha\beta\gamma^2\delta^2)}{16}$$

$$-\frac{[6](\alpha^3\beta^3)+[90](\alpha\beta\gamma^2\delta^2)+[72](\alpha\beta^2\gamma^3)+[24](\alpha^2\beta^2\gamma^2)+[24](\alpha\beta\gamma\delta^3)}{216}$$

$$-\frac{[4](\alpha\beta\gamma\delta^3)+[12](\alpha\beta\gamma^2\delta^2)}{16}\,,$$

where any term, as ex. gr. $[48](\alpha\beta\gamma^2\delta^2)$, means the sum of the quantities of the type $\alpha^2\beta^2\gamma\delta$ each taken a sufficient number of times to make up 48 combinations, so that it is identical in meaning with $8\Sigma(\alpha\beta\gamma^2\delta^2)$ in the common notation. This convention is useful in saving the unnecessary labor of performing divisions in this first part of the process which have to be exactly reversed by multiplications in the transformation process which follows. The value of the above sum is, for purposes of transformation, equivalent to

$$\frac{1}{36}\left\{3\alpha\beta\gamma^2\delta^2+6\alpha\beta^2\gamma^3-4\alpha^2\beta^2\gamma^2-4\alpha\beta\gamma\delta^3-\alpha^3\beta^3\right\},$$

which gives for the image of U

$$\frac{1}{36}(3b^2c^2+6abcd-4ac^3-4db^3-a^2d^2)$$

or $\frac{1}{36}(U-V)$, where it will be observed that $(V-U)$ is identical with the discriminant to $(a, b, c, d \between x, y)^3$. Let us now proceed to find the image of $(U-V)$. Using σ to denote the sum of the combinations of $\alpha, \beta, \gamma, \delta$ taken i and i together, where $\alpha, \beta, \gamma, \delta$ are the roots of the general quartic, we have

$$U-V=\frac{\sigma_1^2\sigma_2^2}{192}+\frac{\sigma_1\sigma_2\sigma_3}{16}-\frac{\sigma_2^3}{54}-\frac{\sigma_3\sigma_1^3}{64}-\frac{\sigma_3^2}{16}$$

$$=\frac{1}{1728}(9\sigma_1^2\sigma_2^2+108\sigma_1\sigma_2\sigma_3-32\sigma_2^3-27\sigma_3\sigma_1^3-108\sigma_3^2).$$

Expanding and transforming, it will be found that the image of $(U - V)$ is $\left(\frac{21}{432}U - \frac{1}{432}V\right)$ and the second image of U which is $\frac{I(U-V)}{36}$ does not revert to the form U.

As a simpler example we may take the covariant to a quartic, still of the fourth order in the coëfficients as before, but of the eighth degree in the variables. This will have for its root-differentiant

$$\lambda a^2(ae - 4bd + 3c^2) + \mu(ac - b^2)^2, \text{ say } \lambda U + \mu V.$$

Here $U = \sigma_4 - \frac{\sigma_1\sigma_3}{4} + \frac{\sigma_2^2}{12} = \frac{1}{12}(12\sigma_4 - 3\sigma_1\sigma_3 + \sigma_2^2)$, and for the purpose of transformation is equivalent to

$$\frac{1}{12}\left\{12\alpha\beta\gamma\delta - 3(4\alpha\beta\gamma\delta + 12\alpha^2\beta\gamma) + 6\alpha\beta\gamma\delta + 6\alpha^2\beta^2 - 12\alpha^2\beta\gamma\right\}$$

$$= \frac{1}{12}\left\{6\alpha\beta\gamma\delta + 6\alpha^2\beta^2 - 12\alpha^2\beta\gamma\right\}.$$

Hence, using I to denote "image of,"

$$IU = \frac{1}{2}\left\{b^4 + a^2c^2 - 2acb^2\right\} = \frac{1}{2}V.$$

Again

$$V = \left(\frac{\sigma_2}{6} - \frac{\sigma_1^2}{16}\right)^2$$

$$= \frac{1}{48^2}\left\{3\alpha^2 + 3\beta^2 + 3\gamma^2 + 3\delta^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)\right\}^2,$$

which, for purposes of transformation will be found equivalent to

$$\frac{1}{48^2}\left\{36\alpha^4 + 132\alpha^2\beta^2 - 48\alpha^2\beta\gamma - 144\alpha^2\beta + 24\alpha\beta\gamma\delta\right\}.$$

Consequently

$$IV = \frac{1}{192}\left\{3a^3e + 11a^2c^2 - 4ab^2c - 12a^2bd + 2b^4\right\}$$

$$= \frac{1}{192}\left\{3a^2(ae - 4bd + 3c^2) + 2(b^2 - ac)^2\right\}$$

$$= \frac{1}{192}(3U + 2V).$$

Let now $\lambda : \mu$ be so chosen that

$$I(\lambda U + \mu V) = \rho(\lambda U + \mu V).$$

This gives

$$\frac{\mu U}{64} + \left(\frac{\lambda}{2} + \frac{\mu}{96}\right) V = \rho\,(\lambda U + \mu V)\,,$$

or

$$\frac{\mu^2}{64} - \frac{\lambda\mu}{96} - \frac{\lambda^2}{2} = 0\,,$$

i. e. $$3\mu^2 - 2\lambda\mu - 96\lambda^2 = 0\,.$$

The two values of $\frac{\mu}{\lambda}$ derived from this equation are 6 and $-\frac{16}{3}$. The corresponding values of ρ will be 6 and $-\frac{1}{12}$. There are thus two definite systems of $\lambda : \mu$, and no more, which will make $\lambda U + \mu V$ self-conjugate and it is obvious that there will be no other values of $\lambda : \mu$ which will make

$$I^2(\lambda U + \mu V) = \rho\,(\lambda U + \mu V),$$

for, $I^2 U$ and $I^2 V$ being determinate linear functions of U, V, we shall have a quadratic equation for determining $\lambda : \mu$, but the two values of $\lambda : \mu$ which make $\lambda U + \mu V$ self-conjugate must satisfy this equation, and hence there can be no others. Reverting to the preceding example of the type [6: 4, 4], we have found

$$IU = \frac{1}{36}U - \frac{1}{36}V$$

$$I(U - V) = \frac{21}{432}U - \frac{1}{432}V.$$

Hence $$IV = -\frac{9}{432}U - \frac{11}{432}V,$$

and making $$I(\lambda U + \mu V) = \rho\,(\lambda U + \mu V)\,,$$

the equation for finding ρ will be

$$\begin{vmatrix} \frac{12}{432} - \rho\;, & -\frac{12}{432} \\ -\frac{9}{432}\;, & -\frac{11}{432} - \rho \end{vmatrix} = 0\,,$$

whence $$\rho_1 = -\frac{1}{27}\,,\quad \rho_2 = \frac{5}{144}\,;$$

also, since

$$\left(\frac{12}{432}\lambda - \frac{9}{432}\mu\right) = \rho\lambda,$$

we shall have

$$\frac{\lambda_1}{\mu_1} = -\frac{9}{28}, \qquad \frac{\lambda_2}{\mu_2} = 3.$$

What intrinsic peculiar properties are possessed by the principal forms* is a question as to which we are at present quite in the dark, as are we also with regard to the general character of the equation in ρ. It were much to be wished that some one would work out the case of a triadelphic type, as for example the type of covariants of the 6th order in the coëfficients and the 6th degree in the variables, to a sextic. It might be supposed from the two preceding examples that the values of ρ are necessarily rational, but it will be shown hereafter that such is not the case.

It is easy to see that the relation between any form belonging to a given type of multiplicity 2 or 3 and its second image may be geometrically represented by means of a quadric curve or surface. Thus suppose the multiplicity is three, and that the three values of ρ are A, B, C. Construct an ellipsoid or hyperboloid whose semiaxes are $\frac{1}{\sqrt{A}}, \frac{1}{\sqrt{B}}, \frac{1}{\sqrt{C}}$. Draw r any radius vector making angles α, β, γ with the principal axes, p a perpendicular from the centre upon the tangent plane at the point where r meets the quadric, making angles λ, μ, ν with these axes. Then if

$$K(\cos\alpha\, U + \cos\beta\, V + \cos\gamma\, W)$$

be any given form of the system for which U, V, W are the principal forms,

$$\frac{K}{pr}(\cos\lambda\, U + \cos\mu\, V + \cos\nu\, W)$$

will be its second image. And we may say that, if a form lies in the direction of the axis of instantaneous rotation, its second image will lie in the perpendicular upon the invariable plane: or more simply if by the direction of a form $\lambda U + \mu V + \nu W$ we understand that of a straight line whose direction cosines are a $\lambda : \mu : \nu$ and by its modulus $\sqrt{\lambda^2 + \mu^2 + \nu^2}$, we may say that if a radius vector of the ellipsoid (or other quadric) represent the direction and modulus of an in- or co-variant the corresponding radius vector of the polar reciprocal to the quadric will represent the direction and modulus of its second image.

* By a principal form (in general), as hereafter stated in the text, I mean one which is the reciprocal of its first image in the sense that it bears a numerical ratio to its second image. The numerical quantity by which it must be multiplied to give the second image, I call a principal multiplier.

The true nature of the reciprocity theorem, in the general case where i, j have any values whatever, is now obvious. Let $U_1, U_2, \ldots . U_q$ be independent forms belonging to the type $[w: i, j]$, whose multiplicity is q, and $V_1, V_2, \ldots . V_q$ as many forms belonging to the reciprocal type $[w: j, i]$. We may, by virtue of the transformation process, express each IU in terms of linear functions of the forms V and *vice versa*, so that each $I^2 U$ will be a known linear function of all the U's. For clearness sake suppose $q = 3$ and let

$$I^2 U_2 = a' U_1 + b' U_2 + c' U_3$$
$$I^2 U_2 = a' U_1 + b' U_2 + c' U_3$$
$$I^2 U_3 = a'' U_1 + b'' U_2 + c'' U_3 .$$

Now make

$$I^2 (\lambda U_1 + \mu U_2 + \nu U_3) = \rho (\lambda U_1 + \mu U_2 + \nu U_3).$$

We shall have for finding ρ the equation

$$\begin{vmatrix} (a - \rho), & b & , c \\ a' & , (b' - \rho), & c' \\ a'' & , b'' & , (c'' - \rho) \end{vmatrix} = O,$$

and then the three systems of values of $\lambda : \mu : \nu$, which makes the second image of $\lambda U_1 + \mu U_2 + \nu U_3$ coincide to a numerical factor près, with itself, will be rational functions of the respective roots. So, in general, when the multiplicity of the type $[w: i, j]$ is q, there will be in general q special forms, and no more, which have reciprocal forms belonging to the type $[w: j, i]$, and if the interchangeable elements, i, j are equal, then these q forms will all be self-conjugate. It is conceivable that in certain cases the equation in ρ may have equal roots; in that event each such equality would introduce a corresponding indeterminateness in the forms admitting of conjugates. For example, if the multiplicity were 2 and the two roots of ρ equal, that would signify that *every* form belonging to the type would have a conjugate—a fact analogous to an ellipse becoming a circle, or an ellipsoid a spheroid—and so in general.

A form having a conjugate, *i. e.* whose second image is a numerical multiplier of itself, may be called a principal form. If the multiplicity of the type is q, there will be q such. All but these will give rise to an endless succession of images such that any $q + 1$ of an even order (the form itself included among these) will be connected by a linear equation. That the succession is endless is clear from the consideration that if an image, say of

the $(2p)$th rank, is identical (to a numerical factor près) with the form, we have an equation of the qth degree for finding the values of the systems of multipliers λ, μ, ν of U, V, W; therefore there are only q such systems, but the systems which satisfy $I^2F = \rho F$ must also satisfy $I^{2p}F = \rho' F$, and consequently there are none others.

To illustrate this, suppose

$$I^2 U = aU + bV$$
$$I^2 V = cU + dV;$$

then

$$I^4 U = (a^2 + bc)\, U + (ab + bd)\, V$$
$$I^4 V = (ca + ad)\, U + (cb + d^2)\, V.$$

If now we put

$$\begin{vmatrix} a - \rho \ , & b \\ c \ , & d - \rho \end{vmatrix} = 0,$$

to find the values of $\lambda : \mu$ which make $I^2(\lambda U + \mu V) = \rho(\lambda U + \mu V)$ we have

$$(a - \rho)\, \lambda + c\mu = 0.$$

In like manner, if we make

$$\begin{vmatrix} a^2 + bc - \rho \ , & ab + bd \\ ca + ad \ , & cb + d^2 - \rho \end{vmatrix} = 0,$$

to find the values of Λ and M which make $I^4(\lambda U + \mu V) = R(\lambda U + \mu V)$, we have

$$(a^2 + bc - R)\, \Lambda + (ca + ad)\, \mathrm{M} = 0,$$

and it will be found that

$$a - \rho = \frac{a - d}{2} \pm \frac{1}{2} \sqrt{(a - d)^2 + 4bc}$$

$$a^2 + bc - R = \frac{a^2 - d^2}{2} \pm \frac{a + d}{2} \sqrt{(a - d)^2 + 4bc}\,,$$

so that the values of $\lambda : \mu$ and $\Lambda : \mathrm{M}$ are the same, and such we know *à priori* must be the case.

It ought to be noticed that the method explained in the preceding pages furnishes a complete solution of the problem following. Given any in- or co-variant, say of the jth order in the coëfficients to a form Q of the ith degree, to find the process of differentiation which performed upon the product

$$Q(x_1, y_1) \,.\, Q(x_2, y_2) \,.\,.\,.\,.\,.\, Q(x_j, y_j)$$

shall produce the j-partite-emanant of the in- or co-variant so given, and it proves incidentally that every binary in- or co-variant may be represented as a hyperdeterminant. To make this clear, let us call the above product,

or rather that product divided by $(\Pi i)^j$, the j-ary norm of Q and denote it by NQ. Again, let G be any given differentiant to the type $[w:j,i]$, say $G(\rho_1, \rho_2 \ldots\ldots \rho_j)$ which is necessarily identical with

$$G\{0;\ (\rho_2-\rho_1);\ (\rho_3-\rho_1);\ \ldots\ldots (\rho_3-1)\}.$$

For $\rho_\kappa - \rho_1$ write $\frac{d}{dx_\kappa}\cdot\frac{d}{dy_1}-\frac{d}{dy_\kappa}\cdot\frac{d}{dx_1}$ and let the quantity so formed be called the hyperdeterminant to G and be denoted by HG. Then if E be any principal form to the type $[w:i,j]$, of the multiplicity q and belonging to a quantic Q, and G be its first image, we shall have

$$(HG)(N_j Q) = \rho F,$$

where ρ is one of the roots of a known equation of the qth degree in ρ. Consequently, since any form belonging to the given type is a linear function of its q principal forms, every such form may be expressed by means of the hyperdeterminant

$$\Sigma_{\lambda=q}^{\lambda=1}\frac{c_\lambda}{\rho_\lambda}(HG_\lambda)\,NQ,$$

the given form being supposed to be expressible by $\Sigma_{\lambda=q}^{\lambda=1} c_\lambda F_\lambda$, where F is any one of the q principal forms.

It follows from what has been shown above that in general from any one particular given form belonging to a type of multiplicity q may be deduced the $(q-1)$ others (by taking the successive second images) and thus the general form obtained; the exception is when the given form happens to be a linear function of less than q of the principal forms. A further consequence is that any in- or co-variant given in terms of the roots of its quantic may be converted by explicit processes into a function of the coëfficients. Thus ex. gr., suppose that the multiplicity of the type is 3; call the given form R_0 and the successive second images R_1, R_2, R_3, R_4. These latter will be all known by the rule of transformation and we shall have R_4 a known linear function of the three preceding forms, say equal to

$$\alpha R_1 + \beta R_2 + \gamma R_3.$$

Hence if we put

$$R_0 = \lambda R_1 + \mu R_2 + \nu R_3,$$

we must have

$$R_1 = \lambda R_2 + \mu R_3 + \nu(\alpha R_1 + \beta R_2 + \gamma R_3);$$

hence

$$\nu = \frac{1}{\alpha},\quad \mu = -\frac{\gamma}{\alpha},\quad \lambda = -\frac{\beta}{\alpha}$$

and thus R_0, given in terms of the roots, become known in terms of the coefficients of its quantic. And so in general, q being the multiplicity, $(q+1)$ forms deduced from the given function of the roots will serve to determine its value as a function of the coëfficients. In fact by regarding R_0 as a linear function of the principal forms, it is easy to see it and all its successive secondaries (*i. e* second images) form a recurring series, the scale of relations being.

$$R_0 - \Sigma \frac{1}{\rho} R_1 + \Sigma \frac{1}{\rho^2} R_2 - \Sigma \frac{1}{\rho^3} R_3 + \ldots\ldots = 0,$$

where $1:\rho$ is the ratio of any principal form to its immediate secondary. Thus E_0 being given in terms of the roots and consequently $E_1, E_2, \ldots\ldots E_q$, in terms of the coëfficients, E_0 becomes known in terms of the coëfficients and of the quantities $\Sigma \frac{1}{\rho}, \Sigma \frac{1}{\rho^2}, \ldots\ldots$; these latter are identical with the quantities previously mentioned and furnish the simplest means of forming the equation in ρ, which (if we agree to call $\rho_1, \rho_2, \ldots\ldots \rho_q$ the moduli of the several principal forms $F_1, F_2, \ldots\ldots F_q$, *i. e.* the ratios of their respective second images to themselves) may be termed the modular equation for any given type.*

It might have been useful, had I thought of it in time, and may be useful when the subject comes again under consideration, to treat a form and its second image, in which the type is restored as *antecedent* and *consequent*, and to describe the first image as the *alternate* form to the primitive, inasmuch as we pass, by what biologists alternate generation, from one type to the other. It has been shown, in what precedes, that the transformation by images at each second step leads back to the original type, but, contrary to what might have been supposed, does not in general imply the resuscitation of the individual form.

The theorem of reciprocity has been seen to be, in its essence, a theorem of differentiants, and ought therefore to admit of being proved by means of the necessary and sufficient partial differential equation to which differentiants are subject. This may be done as follows. If we call $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots\ldots \varepsilon_r$ the successive elements to a binary quantic expressed in its customary form, so that ε_r is the coëfficient of the term containing y_r divested of its numerical binomial coëfficient, and if we write

$$U = \frac{d}{d\alpha} + \frac{d}{d\beta} + \frac{d}{d\gamma} + \ldots\ldots,$$

* But it will be better to adhere to the previous convention and to designate the ρ's as the principal multipliers and the equation in ρ as the principal equation.

where $\alpha, \beta, \gamma, \ldots$ are the roots of the quantic, it is very easily proved that

$$U\varepsilon_r = -r\varepsilon_{r-1}.^*$$

Let $C\Sigma\alpha^r\beta^s\gamma^t \ldots$ be any term in a given differentiant F, the indices $r, s, t, \ldots$ being any whatever with no condition as to their being distinct from each other, and let $N(r, s, t, \ldots)$ signify the number of combinations comprised in Σ; also let $CN(r, s, t, \ldots) \cdot \varepsilon_r \varepsilon_s \varepsilon_t \ldots$ be called the image of the term above written and G the image of F, *i. e.* the sum of the images of the several terms in F; where it must be observed that the ε quantities do not necessarily refer to roots the same in number or name as the roots $\alpha, \beta, \gamma, \ldots$ Now suppose that we have any term, such as $Q\Sigma\alpha^l\beta^m\gamma^n \ldots$ in UF, where U refers to the given roots $\alpha, \beta, \gamma, \ldots$ and means $\frac{d}{d\alpha} + \frac{d}{d\beta} + \frac{d}{d\gamma} + \ldots$ This term must arise from terms of the several forms

$$\left.\begin{array}{l} A\,\Sigma\alpha^{l+1}\,\beta^m\,\gamma^n \ldots \\ B\,\Sigma\alpha^l\,\beta^{m+1}\,\gamma^n \ldots \\ C\,\Sigma\alpha^l\,\beta^m\,\gamma^{n+1} \ldots \end{array}\right\} \text{ in } F,$$
$$\text{etc. etc.} \ldots$$

corresponding to these there will be the images

$$\left.\begin{array}{l} AN\,(l+1, m, n, \ldots)\,\varepsilon^{l+1}.\,\varepsilon^m\,.\,\varepsilon^n \ldots \\ BN\,(l, m+1, n, \ldots)\,\varepsilon_l\,.\,\varepsilon^{m+1}.\,\varepsilon^n \ldots \\ CN\,(l, m, n+1, \ldots)\,\varepsilon_l\,.\,\varepsilon^m\,.\,\varepsilon^{n+1} \ldots \end{array}\right\} \text{ in } G,$$
$$\text{etc. etc.} \ldots$$

where G belongs to a quantic whose type is reciprocal to that of F, and it is clear that the effect of operating upon F with U will be to give

$$Q = A\rho N\,(l+1, m, n, \ldots)(l+1) + B\rho N\,(l, m+1, n, \ldots)(m+1) + C\rho N\,(l, m, n+1, \ldots)(n+1) + \text{etc.} \ldots,$$

ρ being a number easily determinable, but which there is no occasion to express. Again if $R\varepsilon_l.\,\varepsilon_m.\,\varepsilon_n \ldots$ be the correlative term in G, we have by virtue of the formula $U\varepsilon_r = -r\varepsilon_{r-1}$, where the operator U refers to the roots of the quantic of reciprocal type,

$$(-)^w R = AN\,(l+1, m, n, \ldots)(l+1) + BN\,(l, m+1, n, \ldots)(m+1) + CN\,(l, m, n+1, \ldots)(n+1) + \text{etc.} \ldots$$

Consequently, since on account of the identity $F = 0$, we must have $Q = 0$

* In fact it may easily be proved by the ordinary rule for the change of one system of independent variables into another that, if $a_1, a_2, \ldots a_i$ be the roots of $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots \varepsilon^i \between x, y)^i$,

$$\Sigma \frac{d}{da} = -\Sigma_{q=0}^{q=1} q\varepsilon_{q-1} \frac{d}{d\varepsilon_q}.$$

for every term $Q\Sigma\alpha^l.\beta^m.\gamma^n\ldots.$, we must also have $R=\rho^{-1}Q=0$ and therefore, this being true for all the arguments $\varepsilon_l.\varepsilon_m.\varepsilon_n.\ldots$, we must have $UG=0$. Hence, when any quantity F, is a differentiant of a given quantic, its image (as defined in the text) is also a differentiant to a quantic of reciprocal type to the given one. This is the simplest method of establishing the theorem, but still the method originally employed in the note is valuable as serving to establish the important proposition that every in- or co-variant of a binary quantic is a hyperdeterminant.

I will proceed to show that for a system of two or more quantics of degrees $i, i', i'', \ldots.$, we may pass from a covariant of the type $[w: i, j; i', j'; i'', j''; \ldots]$ to one of the type $[w: j, i; i', j'; i'', j''; \ldots]$ by taking its image in respect to the quantic whose indices, i, j, are to be interchanged precisely according to the same rule as if there were no other quantic present. As regards the law of reciprocity, a combination of quantics is analogous to a mixture of gases, according to Dalton's view, each playing the part, as it were, of a vacuum in respect to the other.

Let $[w: i, j; i', j'; \ldots.]$ be the type, $[w: j, i; i' j',; \ldots.]$ one of the antitypes, $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots \varepsilon_j \between x, y)^j$ the general form of the jth degree, $\alpha, \beta, \gamma, \ldots$, its roots when $\varepsilon_0=1$. Let $\eta_r=(-)^r\varepsilon_r$; then, since

$$\Sigma\frac{d}{d\alpha}\varepsilon_r=-r\varepsilon_{r-1}$$

$$\Sigma\frac{d}{d\alpha}\eta_r=r\eta_{r-1}\ .$$

Let D be any differentiant of the given type, $a, b, c, \ldots$ the roots of the quantic of degree i, $a', b', c', \ldots$ the roots of the quantic of degree i', with the usual convention as to the leading coëfficients becoming unities. Let $\Sigma a^l b^m \ldots.$, $\Sigma a'^{l'} b'^{m'} \ldots. \Sigma \ldots.$ be the arguments of any term in

$$\left(\Sigma\frac{d}{da}+\Sigma\frac{d}{da'}+\ldots.\right)D,$$

say UD, then the coëfficient of the term last written will arise from operating with U upon

$$\left.\begin{array}{lllll}
A\cdot\Sigma a^{l+1}. & b^m & \ldots.\ \Sigma a'^{l'} & b'^{m'} & .\ \text{etc.}\ldots \\
+B\cdot\Sigma a^l & .\ b^{m+1} & \ldots.\ \Sigma a'^{l'} & b'^{m'} & .\ \text{etc.}\ldots \\
+\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
+A'\,\Sigma a^l & .\ b^m & \ldots.\ \Sigma a'^{l'+1} & b'^{m'} & .\ \text{etc.}\ldots \\
+B'\,\Sigma a^l & .\ b^m & \ldots.\ \Sigma a'^{l'} & b'^{m'+1} & .\ \text{etc.}\ldots \\
 & \text{etc.}\ldots. & & & \text{etc.}\ldots
\end{array}\right\}$$

and the value of the coëfficient will be

$$\left.\begin{array}{l} A\ (l+1)\ N\ (l+1, m, \ldots.)\ N\ (l', m', \ldots\ldots)\ \ldots \\ +\ B\ (m+1)\ N\ (l, m+1, \ldots.)\ N\ (l', m', \ldots\ldots)\ \ldots \\ +\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ +\ A'\ (l'+1)\ N\ (l, m, \ldots\ldots\ldots)\ N\ (l'+1, m', \ldots)\ \ldots \\ +\ B'\ (m'+1)\ N\ (l, m, \ldots\ldots\ldots)\ N\ (l', m'+1, \ldots)\ \ldots \\ +\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}$$

$$\div\ N\ (l, m, \ldots.)\ N\ (l', m', \ldots)\ \ldots$$

To these feeders or contributory terms will correspond, in the image,

$$\begin{array}{l} A N\ (l+1, m, \ldots.)\ \eta_{l+1} . \eta_m\ \ldots.\ \Sigma a'^{l'}\ .\ b'^{m'}\ \ldots \\ +\ B N\ (l, m+1, \ldots.)\ \eta_l\ .\ \eta_{m+1} . \ldots\ \Sigma a'^{l'}\ .\ b'^{m'}\ \ldots \\ +\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ +\ A' N\ (l, m, \ldots\ldots\ldots)\ \eta_l\ .\ \eta_m\ \ldots.\ \Sigma a'^{l'+1} .\ b'^{m'}\ \ldots \\ +\ B' N\ (l, m, \ldots\ldots\ldots)\ \eta_l\ .\ \eta_m\ \ldots.\ \Sigma a'^{l'}\ .\ b'^{m'+1} . \ldots \\ +\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

and it is obvious that by operating upon this with the U corresponding to its roots we shall obtain the argument $\eta_l . \eta_m . \ldots \Sigma a'^{l'} . b'^{m'}$. etc. . . . affected with the very same coëfficient as that above written, except that in its denominator the factor, $N\ (l, m, \ldots.)$, will not appear. Hence, when D is a differentiant of the given type, its image (obtained by expressing the i set of coëfficients in terms of roots and then replacing every power, ρ^q, of any such root, ρ, by η_q, leaving all the other coëfficients unchanged) will also be a differentiant of the type transformed by interchanging i with its conjugate j.*

When there is but one quantic the effect of substituting ε_q instead of η_q will evidently only be to introduce a common factor $(-)^w$ into each term, which is immaterial and we may accordingly in that case reflect ρ^q into ε_q. Of course, in the general case, if all the letters i are simultaneously interchanged with the letters j, a similar conclusiou follows.

* Thus the rule of images for passing from a differentiant of a given type belonging to a single quantic to one of the opposite type is extended to the case of passing from a differentiant of a given type belonging to a system of quantics to any associated type, *i. e.* to any type in which one or more of the numbers i chosen at discretion is or are interchanged with the corresponding numbers j, and it will presently be seen that this implies the extension of the rule without any alteration from differentiants or invariants to covariants of a quantic or system of quantics. In Note A it will further be shown that for any inversions whatever (or, to speak more accurately, for any cycle of inversions leading back to the original type), although the principal multipliers change their values as the cycle of inversion changes, the *principal forms themselves remain the same*,—a surprising conclusion but very easily proved. In other words, however many quantics there may be in the parent system, there is never more than one single set of principal forms of derivatives to it of a given type. A cycle of arbitrarily intercalated pairs of reversals (here of successive i's and j's), by which a type returns to itself, comes under the category, "Verschlingung," or "Knotting" of Gauss, Listing and Tait.

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W. W. Hendrickson, . .	" "
J. M. Rice, ...	" "
B. Sestini, S. J., . . .	Woodstock Col., Md.
Alexander Evans, . . .	Elkton, "
William H. Parker, . .	Md. Agr. Col., Md.
General Benjamin Alvord, .	Washington, D. C.
Simon Newcomb, . . .	" "
J. H. C. Coffin, . . .	" "
Marcus Baker, . . .	" "
D. P. Todd, ...	Washington, D. C.
J. O. Wiessner, . . .	" "
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Edward B. Smith, . . .	Richmond, Va.
Harry Estill, ...	Ashland, Va.
M. S. Hudgins, . . .	Matthews C. H., Va.
James M. Ingalls, . . .	Morgantown, W.Va.
H. T. J. Ludwig, . . .	Mt. Pleasant, N. C.
Vanderbilt University, . .	Nashville, Tenn.
William J. Vaughn, . .	Franklin, Tenn.
A. H. Buchanan, . . .	Lebanon, "
A. B. Nelson, ...	Danville, Ky.
H. T. Eddy, ...	Cincinnati, O.
Ormond Stone, . . .	" "
E. J. Landor, ...	Canton, "
Marietta College, . . .	Marietta, "
D. J. Herron, . . .	Lafayette, Ind.
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R. Greenwood, . . .	Morris, Ill.
S. W. Shattuch, . . .	Champaign, Ill.
R. J. Adcock, . . .	Monmouth, "
Washington University, . .	St. Louis, Mo.
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W. W. Beman, . . .	" "
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As an example, let us take the two quadratics,

$$ax^2 + 2bxy + cy^2,$$
$$\alpha x^2 + 2\beta xy + \gamma y^2,$$

their resultant, $(a\gamma - c\alpha)^2 + 4(a\beta - b\alpha)(c\beta - b\alpha)$, belongs to the type $[4: 2, 2; 2, 2]$ which is its own reciprocal whichever of the interchangeable elements we permute. This resultant, treating a as unity, will be equal to

$$(\alpha\rho_1^2 + 2\beta\rho_1 + \gamma)(\alpha\rho_2^2 + 2\beta\rho_2 + \gamma)$$
$$= \alpha^2\rho_1^2\rho_2^2 + 2\beta\alpha\,(\rho_1^2\rho_2 + \rho_1\rho_2^2) + 4\beta^2\rho_1\rho_2 + \alpha\gamma\,(\rho_1^2 + \rho_2^2) + 2\beta\gamma\,(\rho_1 + \rho_2) + \gamma^2$$

the image of which will be

$$\alpha^2\varepsilon_2^2 - 4\alpha\beta\varepsilon_1\varepsilon_2 + 4\beta^2\varepsilon_1^2 + 2\alpha\gamma\varepsilon_0\varepsilon_2 - 4\beta\gamma\varepsilon_0\varepsilon_1 + \gamma^2\varepsilon_0^2\,,$$

or as we may write it,

$$\alpha^2c^2 - 4\alpha\beta bc + 4\beta^2b^2 + 2\alpha\gamma ac - 4\beta\gamma ab + a^2\gamma^2\,,$$

which is $(c\alpha - 2b\beta + a\gamma)^2$, the square of the well known connective. Again, if we combine $ax^3 + 3bx^2y + 3cxy^2 + dy^3$ with $\alpha x + \beta y$, we have the invariant

$$a\beta^3 - 3b\alpha\beta^2 + 3c\alpha^2\beta - d\alpha^3\,,\quad \text{say } I,$$

belonging to the type $[3: 3, 1; 1, 3]$. Write $\alpha = 1$, $\beta = -\rho$; this becomes

$$-a\rho^3 - 3b\rho^2 - 3c\rho - d\,,$$

of which an image, say J, belonging to the type $[3: 3, 1; 3, 1]$,

$$a\varepsilon_3 - 3b\varepsilon_2 + 3c\varepsilon_1 - d\varepsilon_0$$

is the connective of $\left\{\begin{array}{l} ax^3 + 3bx^2y + 3cxy^2 + dy^3 \\ \varepsilon_0x^3 + 3\varepsilon_1x^2y + 3\varepsilon_2xy^2 + \varepsilon_3y_3. \end{array}\right\}$

Similarly

$$(a^2d - 3abc + 2b^3)\,\beta^3 \,\ldots\ldots + \ldots\ldots + (d^2a - 3dbc + 2c^3)\alpha^3,$$

say I, belonging to the type $[6: 3, 3; 1, 3]$, will have for a reciprocal

$$(a^2d - 3abc + 2b^3)\,\varepsilon_3 + \ldots\ldots\,(d^2a - 3dbc + 2c^3)\,\varepsilon_0,$$

say J, belonging to the type $[6: 3, 3; 3, 1]$. The graph of I will be that of Fig. 41 and the graph of J, that of Fig. 42, where I use B and G (the initials of boron and gold, instead of Au for the latter) and H (the initial of hydrogen) to represent the algebraical atoms (*i. e.* quantics) of valences (*i. e.* degrees) 3, 3, and 1 respectively. Prefixing Σ to the I graph and substituting G_1, G_2, G_3, the three roots of G, for H, H', H'' and B_1, B_2, B_3 for B, B', B'' we obtain

$$\Sigma\,(B_1 - B_2)(B_1 - B_3)(B_2 - B_3)(B_1 - G_1)(B_2 - G_2)(B_3 - G_3),$$

which by inspection is the root representative of J, and prefixing Σ to the J graph and substituting H for G, we obtain in like manner

$$\Sigma\,(B_1 - B_2)^2(B_2 - B_3)(H - B_1)(H - B_3)^2,$$

as the root representative of I.

It may be observed that Fig. 43 is, algebraically speaking, a pseudograph of J, for its reading would give zero for the value of I.

It follows as an immediate consequence from the preceding extension of the law of images to quantic-systems, that the rule for deducing the first term of the reciprocal to a covariant from that of the covariant itself by writing η_r for a^r holds good as a rule for deducing each term of the one from the corresponding term of the other. To see this we have only to recall that every covariant to a quantic or quantic system may be regarded as an invariant of a new system containing the given quantic or system augmented by a linear quantic whose coëfficients are y and $-x$.

NOTE A TO APPENDIX 2.

Completion of the Theory of Principal Forms.

IN the case of a derivative from a system of k parent quantics, it at first sight would seem that since reversion (the act of forming the second image, or process, as we may term it, of double reflexion) may be effected in regard to each system of coëfficients separately, the method in the text ought to furnish in general k distinct systems of principal forms, but this is a mere mirage of the understanding which disappears as soon as the question is submitted to close examination. There is always an unique set of μ forms (μ being the multiplicity of the type) which revert unchanged (barring a numerical multiplier) whichever system of coëfficients undergoes double reflexion. But a caution is necessary for the right interpretation of this statement. $U, V, W \ldots$ may be the principal forms in regard to one set of coëfficients, $\lambda U + \mu V, W \ldots$, or $\lambda U + \mu V + \nu W \ldots$, where λ, μ, ν are indeterminate, in regard to some other. In any such case we may still say that $U, V, W \ldots$ is the principal system in regard to both sets and so in general. We have an example of this if we take any covariant to a single quantic Q and translate it into an invariant of Q and a linear form L. If $U, V, W \ldots$ are principal forms in respect to Q, $\lambda U + \mu V + \nu W + \ldots$ (*i. e.* the absolutely general form of the type) may be easily shown to undergo reversion in respect to L unaltered. $U, V, W \ldots$ may consequently still be seen to be a principal form system in respect to Q and L, as each of these quantities is unaltered by reversion in respect either to Q or to L.

Suppose now a diadelphic system of which U, V are the principal forms quâ one set of coëfficients. Let R denote a reversion quâ this set, R' quâ some other set. Let $RU = aU$, $RV = bV$ and suppose $R'U = \alpha U + \beta V$. Then $R'RU = a\alpha U + b\beta V$ and $RR'U = a\alpha U + b\beta V$.

But by the nature of the process of reversion $RR' = R'R$; hence $a\beta = b\beta$. If $a = b$, every linear combination of U, V is a principal form quâ R. Hence the principal form quâ the R' set, is such for both sets. But if a is not equal to b, we must have $\beta = 0$. Hence U will be a principal form quâ R' as well as R, and the same will be true of V. For if

$$\begin{aligned} R'V &= \gamma U + \delta V \\ RR'V &= a\gamma U + b\delta V \\ R'RV &= R'bV = b\gamma U + b\delta V. \end{aligned}$$

Therefore $a\gamma = b\gamma$ and $\gamma = 0$. Thus U, V will each of them be common as principal forms to each set. I have gone through the same somewhat tedious process of proof for triadelphic forms and with the same result. The very beautiful conclusion follows that whatever the multiplicity of a type may be and whatever number of sets of coëfficients it involves, there is always *a single system of principal forms* common to all the sets.*

NOTE B TO APPENDIX 2.

Additional Illustrations of the Law of Reciprocity.

ACETIC aldehyde contains two atoms of carbon, one of oxygen and four of hydrogen.† It thus corresponds to the quartic covariant of a quadratic and

*Suppose there are k quantics in the parent system and that a derivative type μ is given. Each simple inversion of a pair of permutable indices (i, j) will give rise to a distinct principal equation; there will therefore be k such equations. Let ρ be a root of one of these, σ a root of any other. Then a principal form may be expressed as a linear function of any μ independent special forms connected by coëfficients which are rational integer functions of ρ. Hence σ may be found as a rational function of ρ; but in like manner ρ may be found as a rational function of σ. Hence ρ, σ must be related by an equation of the form

$$A\rho\sigma + B\rho + C\sigma + D = 0,$$

and thus we see that all the k principal equations are homographically related, *i. e.* that each may be obtained from any other by a substitution of the form

$$\rho = \frac{C\sigma + D}{A\sigma + B}.$$

In a word, the multiplicity μ (whatever the *diversity* k) determines the number of principal forms; and the k sets of principal multipliers are given by k algebraical equations of the μth degree, homographically transformable into one another.

†I originally took chloral as the subject of this investigation, being interested in examining its algebraical constitution in consequence of having had personal experience of its use as an escharotic. But for greater simplicity I have substituted acetic-aldehyde of which chloral is a third emanant, three hydrogen atoms of the former being replaced by three of chlorine in the latter.

quartic, linear and quadratic in respect to the coëfficients of the first and second respectively; such a form exists algebraically (Higher Algebra 3d ed., p. 200) and may easily be proved to be monadelphic. Let us treat it as an invariant: if we were to take for its graph a triangle of which C, C, O were the apices and attach two atoms of hydrogen to each C, the permutation-sum of the product of the differences of the connected letters is zero; this then is a pseudograph. A true graph of it is given by the figure,

$$\begin{array}{c} H\cdot C\cdot O\cdot H \\ : \\ H\cdot C\cdot H \end{array},$$

where each single dot between two letters means a single bond and the two dots between the upper and lower C's stand for a pair of bonds between them. This belongs to the invariantive type [4, 2; 2, 1; 1, 4: 0], the complete reciprocal to which is [2, 4; 1, 2; 4, 1: 0]. The constitution of the latter in terms of the roots is found from the above graph by writing O for C, C for H and H for O and is accordingly

$$\Sigma\,(O-O')^2\,(O-C)\,(O-O')\,(O'-C'')\,(O'-H)\,(H-C),$$

where the factor $(O-O')^2$ may be put outside the sign of summation. We may therefore take for its graph a detached molecule of oxygen + a molecule of formic acid, which latter contains two of oxygen, one of carbon and two of hydrogen.

$$\begin{array}{c} H\cdot C\cdot O\cdot H \\ : \\ O \end{array}$$

will be a graph of it, from which, turning O into C, H into O and C into H we obtain

$$\Sigma\,(C-O')^2\,(C''-H)\,(C'''-O')\,(C'''-H)\,(H-O)$$

as the value, in terms of its roots, of the algebraical equivalent to acetic aldehyde. The graph for formic acid, it may be noticed, exists algebraically (Higher Algebra, p. 300).

Instead of the dissociated molecules of oxygen and formic acid, we may exhibit them combined in the graph

$$\begin{array}{l} C\cdot O\cdot O\cdot O\cdot H \\ : \\ O \end{array}$$

which will give another form to the value of the reciprocal in question, viz.

$$\Sigma\,(C-H)^2(H-O)(H-C')(C'-C'')(C''-C''')(C'''-O)$$

which, not being zero and the type being monadelphic,* must be in a pure numerical ratio to the sum above written.

Chemistry has the same quickening and suggestive influence upon the algebraist as a visit to the Royal Academy, or the old masters may be supposed to have on a Browning or a Tennyson. Indeed it seems to me that an exact homology exists between painting and poetry on the one hand and modern chemistry and modern algebra on the other. In poetry and algebra we have the pure idea elaborated and expressed through the vehicle of language, in painting and chemistry the idea enveloped in matter, depending in part on manual processes and the resources of art for its due manifestation.

A peculiar case might possibly arise in applying the theory of principal forms to a self-reciprocal type $[w : i, i]$ which it is proper to mention. For greater simplicity suppose the type to be diadelphic and let M, N be forms of the type which satisfy the equations

$$IM = \rho M\,, \quad IN = \rho' N\,;$$

the M and N have tacitly been defined to be the principal forms for such a type. Now in general this definition merges into and is coincident with the definition of principal forms for the general case, viz., that I^2M and I^2N must be multiples of M and N and the latter condition might be substituted for the former. But this is not always true, for if $\rho + \rho' = 0$, we shall have

$$I^2M = \rho^2 M\,, \quad I^2N = \rho^2 N\,,$$

and consequently,

$$I^2(M + \lambda N) = \rho^2 (M + \lambda N)\,,$$

so that if we were to follow the general definition the principal forms might become indeterminate, whereas by following the definition special to the self-reciprocal case they are determinate. Thus ex. gr., suppose that P, Q, two particular forms of the type, satisfy the equations

$$IP = \rho Q\,, \quad IQ = \sigma P\,;$$

---

* As an exercise the reader may satisfy himself that this type is monadelphic by the direct application of the rule for finding the multiplicity. It corresponds to a quadratic covariant of the type [2, 4 ; 4, 1 : 2], which is the same (introducing the weight $\frac{2 \cdot 4 + 4 \cdot 1 - 2}{2}$ in lieu of the degree) as the type [5: 2, 4 ; 4, 1] and has the same multiplicity μ by the law of reciprocity as the type [5: 4, 2; 4, 1], viz. the difference between the number of modes of composing 5 and of composiog 4 with two of the numbers 0, 1, 2, 3, 4 and with one of a *distinct* set of the same numbers. The arrangements for the weight 5 will be 4. 1 : 0, 4. 0: 1, 3. 2 : 0, 3. 1 : 1, 3. 0 : 2, 2. 2 : 1, 2. 1 : 2, 2. 0 : 3, 1. 1 : 3, 1. 0 : 4, and for the weight 4, 4. 0 : 0, 3. 1 : 0, 3. 0 : 1, 2. 2 : 0, 2. 1 : 1, 2. 0 : 2, 1. 1 : 2, 1. 0 : 3, 0. 0 : 4. The numbers of the combinations in the two sets of arrangements are respectively 10 and 9. Hence $\mu = 10 - 9 = 1$, or the type is monadelphic. The same result of course follows from the known fundamental scale for a quadro-biquadratic system.

the principal forms will then be

$$\sqrt{\sigma}\, P + \sqrt{\rho}\, Q \text{ and } \sqrt{\sigma}\, P - \sqrt{\rho}\, Q,$$

and the two principal multipliers become $\sqrt{\rho\sigma}$ and $-\sqrt{\rho\sigma}$, so that the principal forms according to the general definition would be indeterminate, but according to the definition proper to self-reciprocal forms strictly determinate.

Let us, as a final example of self-reciprocal type, consider the type [10: 5, 5] which is the same as [5, 5: 5] and corresponds to the covariant of the 5th order in the coëfficients and of the 5th degree in the variables to a quintic. This is diadelphic, as may be found by consulting the table of irreducible forms for the quintic, which will show that it can arise only from the multiplication of the parent quantic itself by its quartinvariant or from that of the quadratic quadri-covariant by the cubic cubo-covariant or from a linear combination of the two products. But without this, the same conclusion may be arrived at by direct calculation of the value of (10: 5, 5) — (9: 5, 5) and the multiplicity will be found to be 18 — 16, or 2 as premised. Let us take as our special forms,

$$P = (ae - 4bd + 3c^2)\,(ace + 2bcd - ad^2 - c^3 - b^2e),$$

$$Q = a\,(a^2f^2 - 10abef + 4acdf + 16ace^2 - 12ad^2e + 16b^2df + 9b^2e^2 - 12bc^2f$$
$$- 76\,bcde + 48\,bd^3 + 48c^3e - 32c^2d^2),$$

where $\frac{Q}{a}$ is the quartinvariant J given by Salmon, p. 207, (3d ed.), being in fact the discriminant of the quadricovariant whose root-differentiant is $ae - 4bd + 3c^2$. Call $\alpha, \beta, \gamma, \delta, \varepsilon$ the five roots of the quintic and make $a = 1$. Q contains the term f^2 which is the image of $\alpha^5\beta^5$ which can only arise from combinations of the coëfficients into which d, e, f none of them enter. But all the terms of Q contain d, e, or f, moreover P has no term containing f^2, therefore IQ does not contain Q but is simply a multiple of P. Again ce^2, which enters into P, is the image of combinations of the form $\alpha^2\beta^4\gamma^4$, and the only term in Q which can give rise to such combinations is $-32c^2d^2$, or

$$-\frac{32}{10^4}\,(\Sigma\alpha\beta)^2\,(\Sigma\alpha\beta\gamma)^2,$$

and each such combination will have unity for its coëfficient and their number is 30. Hence

$$IQ = -\frac{30\cdot 32}{10000}\,P = -\frac{12}{125}\,P.$$

Again, Q contains $-10bef$, and bef is the image of such root-combinations as $\alpha^5\beta^4\gamma$ (60 in number) the only terms in P capable of producing which are $10bc^3d$ and $-3c^5$ or $\frac{1}{5000}\,\Sigma\alpha\,(\Sigma\alpha\beta)^3\,\Sigma\alpha\beta\gamma \quad - \frac{3}{100000}(\Sigma\alpha\beta)^5$. And bef does not appear in P, hence one part of IP will be

$$\left(\frac{60}{-50000}+\frac{3\cdot 5\cdot 60}{1000000}\right)Q, \text{ or } -\frac{3}{10000}\,Q.$$

Again, ce^2 is the image of such combinations as $\alpha^4\beta^4\gamma^2$ (30 in number) and the only terms in P giving rise to such are $-3c^5-8b^2cd^2+10bc^3d-3c^2d^2$; $-3c^5$ is $-\frac{3}{100000}(\Sigma\alpha\beta)^5$ and will give rise to $-\frac{3\cdot 20\,30}{100000}\,ce^2$ in IP; $-8b^2cd^2$ is $-\frac{8}{25000}(\Sigma\alpha)^2\,(\Sigma\alpha\beta)\,(\Sigma\alpha\beta\gamma)^2$ and will give rise to $-\frac{2\cdot 8\cdot 30}{25000}\,ce^2$ in IP; $10bc^3d$ is $\frac{10}{50000}\,\Sigma\alpha\,(\Sigma\alpha\beta)^3\,\Sigma\alpha\beta\gamma$ and will give rise to $\frac{7\cdot 10\,30}{50000}\,ce^2$ in IP; $-3c^2d^2$ is $-\frac{3}{10000}(\Sigma\alpha\beta)^2\,(\Sigma\alpha\beta\gamma)^2$ and will give rise to $-\frac{3\cdot 30}{10000}\,ce^2$ in IP. Hence the total coëfficient of ce^2 in IP is

$$-\frac{9}{500}-\frac{12}{625}+\frac{21}{500}-\frac{9}{1000}=\frac{-90-96+210-45}{5000}=-\frac{21}{5000},$$

and consequently, since P contains the term ce^2 and Q the term $16ce^2$, if

$$IP=\theta P-\frac{3}{10000}\,Q,$$

$$\theta-\frac{3\cdot 16}{10000}=-\frac{21}{5000}, \text{ so that } \theta=\frac{3}{5000},$$

and therefore

$$IP=\frac{3}{5000}\,P-\frac{3}{10000}\,Q,$$

and thus the equation for finding the principal multipliers ρ is

$$\begin{vmatrix} \frac{3}{5000}-\rho, & -\frac{3}{10000} \\ -\frac{12}{125}, & -\rho \end{vmatrix}=0,$$

or, if $\rho=\frac{3\sigma}{10000}$, $\quad\begin{vmatrix} 2-\sigma, & -1 \\ -320, & -\sigma \end{vmatrix}=0.$

Thus $\sigma^2 - 2\sigma - 320 = 0$, the roots of which are irrational. I have thought it advisable to set out the work in this example with some explicitness in order to remove an impression that might otherwise arise from the examples which precede, that the principal multipliers and consequently the principal forms, for self-reciprocal types, necessarily contain only rational numbers.

The work is very much longer for the case of non-self-reciprocal types. The simplest example of such that presents itself to my mind is that of the sextinvariant of a quartic and the quartinvariant of a sextic, for either of which the type is diadelphic. The discussion of this case forms the subject of the annexed Note, for all the calculations of which I am indebted to the labor and skill of Mr. F. Franklin, Fellow of Johns Hopkins University. For the sake of brevity the steps of the work have been suppressed and only the final results set out.

NOTE C TO APPENDIX 2.

On the Principal Forms of the General Sextinvariant to a Quartic and Quartinvariant to a Sextic.

Let

$$L = (ae - 4bd + 3c^2)^3 = \left[\frac{1}{2^3\cdot 3}\Sigma(\alpha-\beta)^2(\gamma-\delta)^2\right]^3,$$

$$M = \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}^2 = (ace + 2bcd - ad^2 - b^2e - c^3)^2 = \left[\frac{1}{2^4\cdot 3^3}\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\alpha-\gamma)(\beta-\delta)\right]^2,$$

$$P = (ag - 6bf + 15ce - 10d^2)^2 = \left[-\frac{1}{2^4\cdot 3\cdot 5}\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\varepsilon-\phi)^2\right]^2,$$

$$Q^* = \begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix} = \left\{\begin{array}{l} aceg - acf^2 - ad^2g + 2adef \\ - ae^3 - b^2eg + b^2f^2 + 2bcdg \\ - 2bcef - 2bd^2f + 2bde^2 - c^3g \\ + 2c^2df + c^2e^2 - 3cd^2e + d^4 \end{array}\right.$$

$$= \frac{1}{2^5\cdot 3^3\cdot 5^3}\Sigma(\alpha-\beta)^4(\gamma-\delta)^4(\varepsilon-\phi)^4 - \frac{71}{2^{10}\cdot 3^4\cdot 5^4}\left[\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\varepsilon-\phi)^2\right]^2.$$

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*M. Faà de Bruno, in the tables at the end of his "Théorie des Formes Binaires," designates Q and $\Sigma(a-\beta)^4(\gamma-\delta)^4(\varepsilon-\phi)^4$ by the same symbol I_4; a misleading circumstance which gave rise in this instance, and might in others to a large amount of useless labor. As can easily be seen from the above, the true value of $\Sigma(a-\beta)^4(\gamma-\delta)^4(\varepsilon-\phi)^4$ is $120\,(71P + 900\,Q)$ $= 120\,(71a^2g^2 - 852abfg + 3030aceg - 900b^2eg - 2820ad^2g + 1800bcdg - 900c^3g - 900acf^2 + 3456b^2f^2 + 1800adef - 14580bcef + 6720bd^2f + 1800c^2df - 900ae^3 + 1800bde^2 + 16875c^2e^2 - 24000cd^2e + 8000d^4)$. It should also be observed that in the expression for Q (the catalecticant) given in the same table, the signs of the terms $-2bd^2f + 2bde^2$ have been interchanged.

Then

$$IL = \frac{P - 6Q}{2^5 \cdot 3^2}, \qquad IM = \frac{P - 33Q}{6^5},$$

$$IP = \frac{L + 2M}{2^4 \cdot 5}, \qquad IQ = \frac{9L - 142M}{2^6 \cdot 3^2 \cdot 5^3},$$

$$I^2 L = \frac{7614L + 23868M}{2^{11} \cdot 3^6 \cdot 5^3}, \qquad I^2 M = \frac{201L + 2162M}{2^{11} \cdot 3^6 \cdot 5^3}.$$

In order that $\lambda L + \mu M$ shall be a principal form we must have

$$(7614 - 2^{11} \cdot 3^6 \cdot 5^3 \rho)\, \lambda + 201\mu = 0,$$

$$23868\lambda + (2162 - 2^{11} \cdot 3^6 \cdot 5^3 \rho)\, \mu = 0,$$

$$\begin{vmatrix} 7614 - 2^{11} \cdot 3^6 \cdot 5^3 \rho, & 201 \\ 23868, & 2162 - 2^{11} \cdot 3^6 \cdot 5^3 \rho \end{vmatrix} = 0,$$

or, putting $\sigma = 2^8 \cdot 3^6 \cdot 5^3 \rho$,

$$\sigma^2 - 1222\sigma + 182250 = 0,$$

where it may perhaps be worth noticing that the last term is $2 \cdot 3^6 \cdot 5^3$ and the coëfficient of the second term $2 \cdot 13 \cdot 47$. We obtain from this equation

$$\rho = \frac{611 \pm \sqrt{191071}}{2^8 \cdot 3^6 \cdot 5^3}. \text{*}$$

The principal forms in L and M will then be found to be

$$201L + (-2726 + 8\sqrt{191071})\, M, \qquad 201L + (-2726 - 8\sqrt{191071})\, M;$$

and those in P and Q

$$101P + (-11436 + 24\sqrt{191071})\, Q, \qquad 101P + (-11436 - 24\sqrt{191071})\, Q.$$

Or, if we please, the principal forms in the two cases may be taken as the factors of $201L^2 - 5452LM - 23868M^2$ and $101P^2 - 22872PQ + 205200Q^2$ respectively.† The question, what reduced quadratic forms can appear in the theory of diadelphic types, may one day or another become the subject of *à priori* investigation and form a new connecting link between the Calculus of Invariants and the Theory of Numbers. The linear functions of L and M and of P and Q, corresponding to the reduced forms of the above expres-

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* The number under the radical sign is, I believe, a prime number, but I have not within reach the tables necessary for verifying this. Professor Newcomb, by an exceedingly ingenious combination of a table of squares with Crelle's table of multipliers, (a real stroke of genius,) was able to ascertain by an inspection (the work of a few minutes) that 191071, if not a prime number, must contain a factor not greater than a certain moderate sized integer (137 if my memory serves me right) which reduces the trials necessary to be made to a very small compass.

† These are reducible to

$(201, 68, -60800 \between L', M)^2$, $(101, -23, -1089667 \between P', Q)^2$, where $L' = L - 14M$, $P' = P - 113Q$.

sions might perhaps be termed the principal *rational* forms of the two types respectively.

It may be well to notice that if $I^2U = \rho U$, then $I^2 \cdot IU = I \cdot I^2 U = \rho IU$, and consequently the principal forms for two reciprocal types are images respectively of one another, and the principal multipliers are the same for the two systems.

NOTE D TO APPENDIX 2.

On the Probable Relation of the Skew Invariants of Binary Quintics and Sextics to one another and to the Skew Invariant of the same Weight of the Binary Nonic.

THE law of reciprocity extended, as it has already been in these pages, to systems of quantics, admits of an additional important generalization.

We know that Regnault's law of substitution holds good for algebraical forms, and in fact when transferred to the algebraical sphere becomes identical with the method which I believe I was the first to employ (now familiar to algebraists through the use made of it by Professors Clebsch and Gordan) to which I gave the name of emanation, (Faà de Bruno, p. 198).

The principle, stated in chemico-algebraical language, is that in algebraical compound any number of atoms of a given valence may be replaced by the same number of *new* equi-valent atoms. [In algebra it is essential to lay a peculiar stress on the word *new*; for if the substituted atoms should be homonymous with the remaining atoms, there is a possibility of the transformed compound reducing to zero. As for instance in the algebraical compound $ab' - a'b$ (the representative, say, of potassic iodide), if the atom of potassium should be changed into another of iodine, (or *vice versa*), the compound, viewed algebraically, would disappear].

The law of reciprocity as I have previously given it, translated into chemico-algebraical language amounts to saying that the total number of atoms of one kind (say m n-valent of one kind) may be replaced by n m-valent atoms of another kind; but by applying the rule of substitution first and then that of reciprocity we may see that the condition of *totality* may be done away with and the proposition reduced to the simplified form that in any algebraical compound *m n-valent atoms may be replaced by n m-valent ones.* Whether this law has any application in the chemical sphere, I must leave to chemists to determine.

In addition to the well known fact that a quintic possesses an invariant of the 18th order and a sextic, one of the 15th order, having obtained a complete scheme of the irreducible invariants for the binary quantic of the 10th degree, I was put in possession of the new fact that this last form possesses an invariant of the 9th order and consequently that the nonic possesses an invariant of the 10th order.*

Now the weight of each of these skew invariants is the same number 45, and I was thus led to suspect that they coëxisted in virtue of some secret connexion. What that connexion is I think that I am now (very unexpectedly) in a position to explain and to show (with a high degree of probability) how the values of these three invariants may be actually deduced and calculated from one another. This follows as a consequence of the combined laws of reciprocity and substitution otherwise called emanation. For suppose we have an invariant of a quantic of the mth degree, of the order np in the coëfficients. By the principle of emanation we may transform this into an invariant to a system of n quantics, each of the degree m and of the order p in each set of coëfficients, and by the generalized law of reciprocity this may be again transformed into an invariant to a system of n quantics, each of degree p and of the order m in each set of coëfficients. If now finally these n quan-

*I have calculated, with the kind assistance of Mr. Halsted, the expression in its canonical form of the generating fraction to a binary quantic of the 10th degree. The coëfficient of m in this fraction developed, represents the number of parameters in the general invariant of the mth order of the given decadic. Its denominator is

$$(1-t^2)(1-t^4)(1-t^6)^2(1-t^8)\,1-t^9)\,1-t^{10})\,1-t^{14})$$

and its numerator is the rational integer function

$$1+2t^6+\ldots\ldots\quad+2t^{42}+t^{48},$$

the successive coëfficients being
1, 0, 0, 0, 0, 0, 2, 0, 4, 2, 7, 6, 15, 13, 16, 25, 22, 31, 34, 40, 41, 47, 46, 49, 48, 49, 46, 47, 41, 40, 34, 31, 22, 25, 16, 13, 15, 6, 7, 2, 4, 0, 2, 0, 0, 0, 0, 0, 1,
showing that the primary fundamental invariants are of the orders 2, 4, 6, 6, 8, 9, 10, 14, and that (by the law of "Tamisage" *anglice* siftage) the secondary (or as they might be better termed the auxiliary) ones are of the orders 6, 8, 9, 10, 11, 12, 13, 14, 15, 17 taken 2, 4, 2, 7, 6, 12, 13, 18, 21, 11 times respectively. Any other invariant of the decadic can be represented as a linear function of a limited number of combinations of the secondaries, having for its coëfficients some combination of powers of the primaries.

Suppose that the same numerical order occurs among the primaries and secondaries, as ex. gr. 6, which occurs twice among the former and twice among the latter. This will indicate in the first place that, calling A and B the quadric and quartic invariants, the general sextic one will be of the form

$$\lambda A^3+\mu AB+\nu_1 Q_1+\nu_2 Q_2+\nu_3 Q_3+\nu_4 Q_4$$

and that any two independent special values of $\nu_1 Q_1+\nu_2 Q_2+\nu_3 Q_3+\nu_4 Q_4$ may be taken as primaries and any other independent two as secondaries, and so in general; I mention this to prevent the false suggestion, which might otherwise arise, that the secondaries and primaries are different in internal constitution. This remark receives a beautiful illustration in an algebraical theory (recently developed by me) of chemical isomerism, which gives rise to a generating function precisely similar in character to that applicable to in- and covariants and is subject to a similar law of interpretation, graphs taking the place of algebraical forms, and atomicules and the numbers of grouped atoms, of degrees and orders.

tics, be all made identical with one another, then the transformed invariant, *provided it does not vanish*, becomes an invariant of the order mn to a single quantic of the degree p, and accordingly we may pass in certain cases from the type $[m, np: 0]$ to the type $[p, mn: 0]$. So in all probability we may pass from the type $[5, 18: 0]$ to the type $[6, 15: 0]$ and to the type $[9, 10: 0]$. As there is only one invariant of the type $[6, 15: 0]$, or of the type $[9, 10: 0]$, it follows that, if the passage from type to type is real and not nugatory, the three invariants of these second types may be deduced, any one from any other, by the explicit processes above described. There is nothing at all doubtful in the course of the transformation except what arises from the possibility that in the last step of it the effect of rendering identical the different sets of coëfficients—*i. e.* of finding the counter-emanant, so to say, of the invariant containing n sets of variables—may be to render the whole expression null. This of course would happen if we attempted to pass from the type $[5, 18: 0]$ to the type $[3, 30: 0]$, or to the type $[2, 45: 0]$, which we know are void of forms. But there is no reason why we should expect this to happen when we pass from the given type to other types known to contain one or more forms. It would require no impracticable amount of labor to actually verify the fact of the transformation being effectual between the skew invariants of the sextic and quintic forms. The survival of a *single* known term in either of them, in the process of attempting to deduce it from the other, would be sufficient to establish the effectualness of the method, and that it will be found to be effectual, for reasons too long to dwell upon here, I scarcely entertain a doubt. The process to be employed may be summarily comprehended under the three rubrics of diversification, reciprocation and unification. The first is one of differentiation alone; the second involves the expansion of functions of the coëfficients of an equation in terms of roots and the substitution of η_i for α^i; the third consists merely in replacing distinct sets of letters (η) by a single set. In practice the two latter processes would be of course combined into one. It will be instructive to consider some simple example of this method of transformation of types.

Let us take $(ac - b^2)^3$ regarded as belonging to the type $[2, 6: 0]$. I shall show how to pass from this to a form of the type $[3, 4: 0]$. Taking a third emanant of the given form, *i. e.* the result of the operation upon it of $\frac{1}{1\cdot 2\cdot 3}(a'\delta_a + b'\delta_a)^3$, we obtain

$$(ac' + a'c - 2bb')^3 + 2(ac - b^2)(a'b' - b'^2)(ac' + a'c - 2bb').$$

Let us call $\alpha, \beta, \alpha', \beta'$ the roots of the two forms $[1, b, c]$, $[1, b', c']$ respectively; then the emanant last found (multiplied by 8) becomes

$$(2\alpha\beta + 2\alpha'\beta' - \alpha\alpha' - \alpha\beta' - \beta\alpha' - \beta\beta')\left(\overline{2\alpha\beta + 2\alpha'\beta' - \alpha\alpha' - \alpha\beta' - \beta\alpha' - \beta\beta'}^{\,2} + \overline{\alpha - \beta}^{\,2} \cdot \overline{\alpha' - \beta'}^{\,2}\right).$$

After performing all the multiplications and introducing the zero powers of $\alpha, \alpha', \beta, \beta'$ in such terms as do not contain one or more of these letters, all that remains is to substitute

$$\begin{aligned} \alpha^0 = \alpha'^0 = \beta^0 = \beta'^0 &= \quad a, \\ \alpha = \alpha' = \beta = \beta' &= -b, \\ \alpha^2 = \alpha'^2 = \beta^2 = \beta'^2 &= \quad c, \\ \alpha^3 = \alpha'^3 = \beta^3 = \beta'^3 &= -d, \end{aligned}$$

the letters a, b, c, d for greater simplicity being used instead of $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$, *i. e.* $\eta_0, -\eta_1, \eta_2, -\eta_3$. The result will not vanish. To show this consider the group of terms which change into a^2d^2. These are the binary combinations of $\alpha^3, \alpha'^3, \beta^3, \beta'^3$. $2\alpha\beta$ and $2\alpha'\beta'$ in the first factor give rise to $8\alpha^3\beta^3$, $8\alpha'^3\beta'^3$ and the remaining four terms to $-2\alpha^3\alpha'^3, -2\alpha^3\beta'^3, -2\beta^3\alpha'^3, -2\beta^3\beta'^3$ respectively. Hence the term a^2d^2 will survive with the multiplier $8 + 8 - 2 - 2 - 2 - 2$, *i. e.*, 8. So again the only terms introducing ac^3 will be the ternary combinations of $\alpha^2, \alpha'^2, \beta^2, \beta'^2$. $2\alpha\beta$ and $2\alpha'\beta'$ will be found to produce as many positive as negative terms of this kind, but $-\alpha\alpha'$ will produce $4\alpha^2\alpha'^2\beta^2 + 4\alpha^2\beta'^2\beta'^2$, giving rise to $8ac^3$, and as the same will be true for $-\alpha\beta', -\beta\alpha', -\beta\beta'$, we see that $32ac^3$ will emerge in the result. Hence the given invariant becomes converted into

$$(a^2d^2 + 4ac^3 + \ldots\ldots),$$

i. e. the discriminant of the cubic whose type is $[3, 4: 0]$ as was to be shown.

I think it is little doubtful that wherever there exist forms contained under each of two types, the product of whose rank and order is identical, we may pass from the one to the other by means of the combined processes of emanation and reciprocation, as in the foregoing example.* [The case is

* Call $(b^2 - ac)^3 = A$, $a^2d^2 + 4ac^3 + \ldots. = B$, $a'\delta_a + b'\delta_b + c'\delta_c = E$, $a\delta_{a'} + b\delta_{b'} + c\delta_{c'} + d\delta_{c'} = H^{-1}$. Then it follows from the text that

$$B = \tfrac{1}{12} H^{-2} I E^3 A,$$

where it may be observed that E^3A is diadelphic, for it will be proved that $(6: 3, 2; 3, 2) = 16$, and $(5: 3, 2; 3, 2) = 14$, so that any form whatever coming under the same type as E^3A is a linear function of $(ac' + a'c - 2bb')^3$ and $(ac' - a'c - 2bb')(ac - b^2)(a'c' - b'^2)$, say L and M, (whose difference, $L - M$, is $\frac{1}{6}E^3A$), and operated on by $H^{-2}I$ would produce a multiple of B (whose type is monadelphic) with the sole exception of $\lambda L - 2\lambda M$, the result of operating upon which would be zero. Similarly we may see that in any given case the chances are infinitely in favor of the expectation that the process will *not* be nugatory by which it has been shown we may pass from one known type $[m, np: 0]$ to another known one $[p, nm: 0]$.

much the same as with transvection. That process may produce a null form, but any actually existent form may be produced by it and exhibited as a transvect.] To pass from Hermite's to Cayley's skew form, we must first by emanation change [5, 18: 0] into [5, 6; 5, 6; 5, 6: 0] and then this latter into [6, 15, 0]; by means of the process last exemplified.

APPENDIX 3.

ON CLEBSCH'S THEORY OF THE "EINFACHSTES SYSTEM ASSOCIIRTER FORMEN" (*vide Binären Formen, p. 330*) AND ITS GENERALIZATION.

LET $(a, b, c, \ldots . k, l \between x, y)^n$ be any binary quantic. Let the provector symbol $(l\delta_k + 2k\delta_h + 3h\delta_g + \ldots .)$ be denoted by Ω, and the revector symbol $(a\delta_b + 2b\delta_c + 3c\delta_d + \ldots .)$ by $\mho$. Let Q_{2i} represent the quadrinvariant of the above form when $n = 2i$. Now let Ω and $\mho$ be made to comprise the $2i + 1$ letters $a, b, c, \ldots . l, m$; then $a\Omega Q_{2i} - 2bQ_{2i}$* will be nullified by the operation of $\mho$ and will therefore be a cubinvariant for the case of $n = 2i + 1$ which we may call Q_{2i+1}. Also let $Q_0 = a$; then $Q_0, Q_1, Q_2, \ldots . Q_\mu$ will be differentiants to all binary quantics of degree equal to or greater than μ. The above I call basic differentiants. Their distinguishing characteristic is that the highest letter in each of them enters into it only in the first degree multiplied by a or by a^2 and by no other letter. Now let D be any given differentiant of degree μ and for the moment make $a = 1$. Then it is obvious that D may be expressed—by means of successive substitutions of its ultimate, its penultimate, its antepenultimate, etc. letters up to c inclusive, in terms of the corresponding basic differentiants and the anterior letters, — as a rational integer function of $Q_1, Q_2, \ldots . Q_\mu, b$; or, restoring to a its general value, will be a rational integer function of $Q_0, Q_1, Q_2, \ldots . Q_\mu, b$, say F, divided by a power of a. But I say that b will have disappeared in the process. For $\mho D = 0$; and $\mho Q_0 = 0$, $\mho Q_1 = 0 \ldots . \mho Q_\mu = 0$. Hence, regarding each Q as a constant, $\left(a \frac{d}{db}\right) F = 0$, or F does not contain b.

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*For by a well known formula if D is a differentiant in x of the type $[w : i, j]$, $\mho\Omega D = (ij - 2w)\, D$. Consequently when Q_{2i} is regarded as a differentiant in x of the type $[2i: 2i + 1, 2]$ $\mho\Omega Q_{2i} = Q_{2i}$ also $\mho Q_{2i} = 0$ and $\mho b = a$. Hence $\mho\,(a\Omega Q_{2i} - 2bQ_{2i}) = 0$.

Again, suppose we take a system of two quantics and let $Q_0, Q_1, \ldots . Q_\mu$ be the basic differentiants of the one, $Q'_0, Q'_1, \ldots . Q'_\nu$ of the other, and let D be any differentiant of the system. Then by the same method as before we shall find

$$D = \frac{F(Q_0, Q_2 \ldots . Q_\mu : Q'_0, Q'_2 \ldots . Q'_\nu : b, b')}{a^m \cdot a'^n} .$$

Also each Q will be nullified by $\mho$, and each Q' by $\mho'$, and therefore each Q and Q' as well as D will be nullified by the operator $\mho + \mho'$. Hence we shall have

$$\left(a\frac{d}{db} + a'\frac{d}{db'}\right) F = 0,$$

or

$$F = \phi\,(ab' - a'b) ,$$

ϕ being a rational integral form of function. In like manner for a system of three quantics, regarding the several sets of its basic differentiants as constant, we shall have

$$F = \phi\,(ab' - a'b : ac' - a'c : bc' - b'c) ,$$

where ϕ is a rational integral form of function, or

$$F = \psi\,(ab' - a'b : ac' - a'c : a, a') ,$$

and so in general. Hence, remembering that any relation between differentiants must continue to subsist between the covariants of which they are the roots, and now, understanding by base forms the complete covariants of which the basic coëfficients are the roots, we may pass from differentiants to in- or co-variants and obtain the following theorems.

1°. For a single quantic of degree i, any in- or co-variant is expressible by a fraction whose numerator is a rational integer function of its i base forms and whose denominator is a power of the quantic. This is Clebsch's theorem.

2°. For a system of quantics, any in- or co-variant is expressible by a fraction whose numerator is a rational integer function of the separate base forms of its several quantics and of any complete system of $(\mu - 1)$ independent Jacobians of the quantics taken in pairs, and whose denominator is a product of powers of the quantics of the system.

Also it will be observed that these theorems will continue to subsist when the base forms have for their roots in lieu of the basic differentiants, as above

defined, any ascending scale of differentiants in which the letters enter successively one at a time and each letter on its first appearance figures only in the first degree and combined exclusively with powers of a.

On the theory of basic forms may be grounded a method for obtaining, *in propriâ personâ*, the fundamental in- and co-variants to a quantic or system of quantics in regular succession, by a process which continues so long as there are many more to be elicited and comes to a self-manifesting end as soon as the last irreducible form has been obtained, like an air pump that refuses to act as soon as the exhaustion has become complete. In a word, the cataloguing of the irreducible in- and co-variants is transferred to the province of, and becomes a problem in, ordinary algebra.

I have previously observed that any expression which represents a differentiant in regard to a quantic of a given degree necessarily does the same for quantics of all higher degrees. And I may take this occasion to remark, or to repeat, that a differentiant may be irreducible in respect to the quantic of minimum degree to which it can be referred, and yet not so for quantics of higher degrees. Thus, if we take the expression

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd,$$

this referred to a cubic is irreducible (as is well known), but regarded as a differentiant of a quartic or higher degreed quantic, is reducible, being in fact identical with

$$(ac - b^2)(ae - 4bd + 3c^2) - a\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}.$$

Let us suppose a linear function $yu - xv$ combined with a quantic into a system. Then it follows as a corollary from (2° at p. 118), that if the quantic belongs to the form $(a, b, c, \ldots . l \between u, v)^i$, or say more simply to the form $[a, b, c, \ldots . l]$ any covariant of such quantic multiplied by a suitable power of a will be a function of y, $ax + by$ and of the differentiants, or in a word, every covariant of the quantic expressed as a function of x and $ax + by$ will have no coëfficients but what are differentiants, or to use Professor Cayley's term, semi-invariants. Thus, ex. gr., the Hessian of the cubic $(a, b, c, d \between x, y)^3$ may be put under the form

$$\frac{1}{a^2}\left\{(ac - b^2)(ax + by)^2 + (a^2d - 3abc + 2b^3)(ax + by)y + (ac - b^2)^2y^2\right\},$$

So it will be found that the Hessian of the quintic, viz.

$$(ae - 4bc + 3c^2)\,x^2 + (af - 3be + 2cd)\,xy + (bf - 4cd + 3d^2)\,y^2$$

on writing $ax + by = X$ becomes

$$\frac{1}{a^2}\Big\{(ae - 4bc + 3c^2)\,X^2 + (a^2f - 5abe + 2acd + 8b^2d - 6bc^2)\,Xy$$
$$- \Big[(ac - b^2)(ae - 4bd + 3c^2) + 3a\,(ace + 2bcd - ad^2 - b^2e - c^3)\Big]\,y^2\Big\},$$

where all the coefficients are semi-invariants-in-x, the second coefficient being one of the basic differentiants and the latter part of the third coefficient, the catalecticant

$$\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix},$$

and so more generally, it may be shown to follow from (2°), that if there be any number of binary quantics

$$[a, b, c \ldots .],\ [a', b', c', \ldots .],\ [a'', b'', c'' \ldots .],$$

every covariant of such system, expressed as a function of y and of *any one* of the quantics

$$ax + by,\ a'x + b'y,\ \ldots\ldots$$

chosen at will, has differentiants-in-x exclusively for its coefficients.

It is easy to express the base-covariants in terms of the roots. Those of weight $2n$ and order 2 will be of the form

$$\Sigma F\,(a_1, a_2, a_3, \ldots . a_{2n})\,(x - a_{2n+1})^2(x - a_{2n+2})^2 \ldots\ldots$$

where F may be expressed as

$$(a_1 - a_2)^2(a_3 - a_4)^2 \ldots\ldots (a_{2n-1} - a_{2n})^2,$$

or, $$(a_1 - a_2)(a_2 - a_3)(a_3 - a_4) \ldots\ldots (a_{2n-1} - a_{2n})(a_{2n} - a_1),$$

or under a variety of other forms all equal to a numerical factor près; for the type $[2n : 2n, 2]$ and the more general one $[2n : 2n + \nu, 2]$ are monadelphic. And again those of the weight $2n + 1$ and order 3 may take, or at all events be replaced by, the form

$$\Sigma\big(\overline{a_1 - a_2}\,.\,\overline{a_2 - a_3} \ldots \overline{a_{2n-1} - a_{2n}}\,.\,\overline{a_{2n} - a_1}\,.\,\overline{a_1 - a_{2n+1}}\,.\,\overline{x - a_1}\,.\,\overline{x - a_2} \ldots \overline{x - a_{2n+1}}\,.\,\overline{x - a_{2n+2}}^{\,3}.\,\overline{x - a_{2n+3}}^{\,3} \ldots\big)$$

It is proper to notice that the type $[2n + 1 : 2n + 1 + \nu;\ 3]$ is only monadelphic so long as $2n + 1$ is less than 9, so that we cannot, without an investigation which might be tedious, determine whether the above representation coincides with the basic forms of the third order in the coefficients adopted in p. 118; but such investigation would be a work of supererogation, for the only *material* character for any of the base-covariants in question to possess

is, that its root differentiant-in-x shall be not higher than of the third order in the coefficients and shall contain the element ε_{2n+1}. Any formula having this property (which is enjoyed by the root function above given) is just as good as any other for the purposes of this theory.*

It will be seen to follow from the theorem I have given for differentiants from which Clebsch's follows as an immediate consequence, that all the permutation-sums of any rational integer function of the differences of the roots of an algebraical equation of the nth degree are rational integer functions of $(n-1)$ of them of the second and third order alternately; so, for example, all the coefficients in Lagrange's equations to the squares of the differences of the roots of an algebraical equation in its ordinary form are rational integer functions of $(n-1)$ known quantities. Thus, for instance, the equation to the squares of the differences of a cubic equation will be

$$\rho^3 + 18\,(b^2 - ac)\,\rho^2 + 81\,(b^2 - ac)^2 + 27\Delta = 0\,,$$

where the coefficients are given in terms of two differentiants $(b^2 - ac)$ and Δ.

Throughout this paper the perspicuity of expression has been considerably marred by want of a complete nomenclature which the theory of graphs and types necessarily calls for and which I shall hereafter employ whenever I may have occasion to revert to the subject. It is as follows:

In the first place, w, the weight in respect to the selected variable, and j, the order in the coefficients, are terms well understood and need no change or further illustration; i, the degree of the parent quantic, I shall hereafter call the *rank* of the type, $ij - 2w$ which becomes the degree of a covariant got by expanding the differentiant of type $[w : i, j]$ may be called the *grade*. The order and rank may be termed collectively the *permutable indices*.

When a differentiant is given algebraically its weight and order are given but *not* its rank; in addition to the weight and order a third number which may be called the *range* (and which I shall denote by a Greek ε) is

* Writing the type under the form $[2n+1 : 2n+1+\nu, 3]$. the degree of the corresponding covariant in the variables is $2n+1+3\nu$, which is the degree in x of the symmetrical function assumed in the text; also each letter in this function occurs 3 times agreeing with the order 3 of the type, and the number of factors in the coefficient of the highest power of x is $2n+1$, which is right for the weight It is obvious also by inspection that the product $a_1.a_2.\ldots.a_{2n+1}$ will arise from each term of the assumed symbolical function affected always with the same sign, so that ε_{2n+1} will occur (as required) in its expression in terms of the coefficients. Of course all the same conclusions will apply if in the formula $(a_1 - a_2)^2(a_3 - a_4)^2 \ldots (a_{2n-1} - a_{2n})^2$ is substituted in lieu of $(a_1 - a_2)(a_2 - a_3) \ldots (a_{2n-1} - a_{2n})(a_{2n} - a_1)$.

That the type to which Q_{2n+1} belongs is non-monadelphic from and after $2n+1 = 9$ is obvious from the fact that that type, when the degree of the parent quantic is made a minimum, is of the form $[2n+1 : 2n+1, 3]$, the multiplicity of which is the same as that of $[2n+1 : 3, 2n+1]$, or set out in full $[2n+1 : 3, 2n+1 : 2n+1]$; but cubics include covariants of orders and degrees $2:2$ and $3:3$ among their fundamental forms, and $9:9$ can be formed either by taking a triplication of $3:3$, or by combining $3:3$ with a triplication of $2:2$, so that when $2n+1 = 9$ the type is diadelphic, and *a fortiori*, it is non-monadelphic for values of $2n+1$ superior to 9.

given, being the number less 1 of the letters which enter into it. The relation between *rank* and *range* is one of inequality. The former may be equal to, or greater than, but not less than the latter.

The multiplicity of the type to which a given differentiant belongs is a function of the *weight, order* and *rank* and is consequently not known until the *rank* is assigned. Thus, ex. gr. $(ac - b^2)^2$, considered as having the lowest possible rank, viz. 2, (the *range*) is monadelphic; its type is then [2: 2, 4], but if the rank 4 be assigned to it so that its type is [2: 4, 4], it becomes diadelphic. We have then, in general, 6 characters (not all independent) appertaining to a differentiant, viz., *weight, rank, order, grade, range* and *multiplicity*. The theory of types has never hitherto formed the subject of distinct contemplation, and that is why the necessity for the use of some of the above terms has not been previously felt. But it will have been observed that throughout the preceding memoir it has forced itself upon our notice, and in particular, that it is impossible to go to the bottom of the so-called law of reciprocity or that of the radical representation of forms without keeping in view the question of type and multiplicity.

I have also to remark that since the preceding matter was completed I have been surprised to learn that recent chemical research favors the notion of simple elements (hydrogen atoms in special) being distinguishable from each other in chemical composition. If this view is confirmed, the discrepancy, which I have pointed to, between the known conditions for the existence of algebraical graphs and the unknown natural laws which govern the production of chemical substances may become partially or wholly obliterated, so that, for example, the hydrogen molecule and the extended derivatives from marsh gas may exist in accordance with, and not in contradiction to, algebraical law, and thus it is possible to conceive that all the phenomena of chemistry and algebra may ultimately be shown to be identical.

Since the above matter was sent to press I have been led to study algebraically what may be termed the direct problem of isomerism, that is to say the determination of the number of combinations subject to given conditions that can be formed between the constituents of groups each containing a given number of equivalent chemical atoms, the valences of the several groups being either independent or given linear functions of a certain number of independent parameters. In this problem the numbers of atoms are given and the valences left indeterminate. In the inverse problem the valences are given and the numbers left indeterminate.

The problem of the enumeration of the saturated hydro-carbons, investigated by Professor Cayley, is a simple example of the inverse problem. The direct problem admits of a uniform and unfailing method of solution by generating functions, the exposition of which may probably form the subject of an additional Appendix in the following number.* This method is substantially the same as that which I have described in general terms in the *Comptes Rendus* as applicable to the theory of ternary and other higher varieties of quantics but less difficult of application to the Isomeric Problem on account of the greater simplicity of the crude forms† subject to reduction, which appear in it. Appendix 4 will contain the application of the theory of "Associirter Formen" to the algebraical deduction of the irreducible forms of the quintic and certain other cases which but for the press of matter awaiting publication in the Journal would have formed part (as announced) of the present Appendix.

As already stated in a previous foot-note, the theory of irreducible forms reappears in the isomeric investigation, the general character of the reduced generating function to be interpreted in it being precisely the same as in the invariantive theory, which constitutes an additional and a closer and more real bond of connexion between the chemical and algebraical theories than any which I had in view when I commenced the subject of this memoir.

* The principle employed in this method leads to the following theorem only a particular case of which comes into play in the general partition problem which covers the ground occupied by the allied invariantive and isomeric theories. Let there be given a product of a limited number of rational functions of

$$u_1^{a_1} . u_2^{a_2} . \dots u_i^{a_i} ; \; u_1^{a'_1} . u_2^{a'_2} . \dots u_i^{a'_i} ; \text{ etc., etc.,}$$

where all the indices are *positive or negative* integers, and let $\mu_1, \mu_2, \dots \mu_2$ be given linear functions of $v_1 . v_2, \dots v_j$ (j being not greater than i), then it is always possible to find a limited product of rational functions of

$$v_1^{\beta_1} . v_2^{\beta_2} . \dots v_j^{\beta_j} ; \; v_1^{\beta'_1} . v_2^{\beta'_2} . \dots v_j^{\beta'_j} ; \text{ etc., etc.,}$$

where the indices are exclusively *positive*, such that the coefficient of $v_1^{\nu_1} . v_2^{\nu_2} \dots v_j^{\nu_j}$, in their product developed according to ascending powers of $v_1, v_2, \dots v_j$, shall be the same as the coefficient of $u_1^{\mu_1} u_2^{\mu_2} \dots u_i^{\mu_i}$ in the original product developed according to ascending powers of $u_1, u_2, \dots u_i$. Previous to the discovery of this principle the problem of isomerism, now completely solved potentially for the direct case, must have remained unattackable by any existing methods, such for example as were known to Euler, the inventor of the application of the method of generating functions to the theory of partitions. It renders supererogatory a large part of the methods devised by myself for the treatment of the problem of compound partitions contained in the printed notes of my lectures on Partitions, delivered at King's College, London, in the year 1859. As an example of the direct problem of isomerism, suppose that three atoms of the same valence j are to combine with ϵ atoms of hydrogen which do not combine *inter se;* then the number of combinations which can be so formed is the coefficient of $a^j x^\epsilon$ in the development of the generating function $\dfrac{1 + ax + a^2x^2}{(1-a^2)(1-ax)^2(1-ax^3)}$ if the three atoms are all unlike, and of the generating function $\dfrac{1}{(1-a^2)(1-ax)(1-a^2x^2)(1-ax^3)}$ if they are all alike

† Very unluckily printed as "*formes cubiques*" in the Comptes Rendus.

NOTE ON THE LADENBURG CARBON-GRAPH.

The reasoning by which I have established, in the preceding number of the *Journal*, the validity of the Ladenburg graph (and the invalidity of the Kekulean one) as a representative of the root differentiant to a covariant of the 6th degree in the variables and of the 6th order in the coefficients to a quartic, is so peculiar and it may seem to some of my readers so far-fetched, that it appears highly desirable to confirm it by a direct demonstration founded on the principle, that the permutation-sum of the product of the bonds in a valid graph interpreted as differences between the letters which they connect, shall not vanish. Previous to applying this principle to Ladenburg's graph we must convert it into an invariant by attaching hydrogen atoms to the six apices. Let these apices be called a, b, c, d, e, f, and the hydrogen atoms $\alpha, \beta, \gamma, \delta, \varepsilon, \phi$: then the permutation-sum under consideration is

$$\Sigma\,(a-b)(a-c)(b-c)(d-e)(d-f)(e-f)\,(a-d)\,(b-e)\,(e-f)\,(a-\alpha)\,(b-\beta)\,(c-\gamma)\,(d-\delta)\,(e-\varepsilon)\,(f-\phi)$$

where the 6 letters a, b, c, d, e, f are interpermutable, as are also the 6 letters $\alpha, \beta, \gamma, \delta, \varepsilon, \phi$.

It may be well to observe at this point that if we struck off the hydrogen atoms and treated the graph as representing an invariant to a cubic form, the permutation-sum

$$\Sigma\,(a-b)\,(a-c)\,(b-c)\,(d-e)\,(d-f)\,(e-f)\,(a-d)\,(d-c)\,(c-f)$$

would be found to vanish, as may easily be shown and as it ought to do, because there exists no invariant of the 6th order in the coefficients to a cubic form. Let a and d be interchanged in the term given under the sign of summation in the permutation-sum formed from the Ladenburg graph; then the sum of this together with the original term becomes

$$(a-d)\,(b-e)\,(c-f)\,(b-c)\,(e-f)\,(b-\beta)\,(c-\gamma)\,(e-\varepsilon)\,(f-\phi)$$

multiplied by

$$(a\delta-d\alpha)\,(a^2-\overline{b+c}\,a+bc)\,(d^2-\overline{e+f}\,d+ef)-(d\delta-a\alpha)\,(d^2-\overline{b+c}\,d+bc)\,(a^2-\overline{e+f}\,a+ef),$$

which last named multiplier will be found to contain the quantity $(a^3d^2-a^2d^3)\,(\alpha+\delta)$. Again, in the multiplicand, let b and c be interchanged; then, since

$$(b-e)\,(c-f)-(c-e)\,(b-f)=(b-c)\,(e-f),$$

the sum of the original and permuted multiplicand will contain a term

$$(a-d)\,(b-c)^2\,(e-f)^2\,bc\,(e-\varepsilon)\,(f-\phi),$$

and accordingly the entire permutation-sum will contain the terms

$$(\alpha+\delta)\,(a-d)\,(a^3d^2-a^2d^3)\,(b-c)^2(e-f)^2\,bc\Sigma\,(e-\varepsilon)\,(f-\phi).$$

The partial sum last written is

$$4ef+4\varepsilon\phi-2\,(e+f)\,(\varepsilon+\phi).$$

Hence we may readily see that the total permutation-sum will contain *inter alia* a positive multiple of the combination $a^4b^3e^3d^2cf\alpha$ and will not vanish, and consequently the graph is valid and not illusory; I presume that the same method applied to Kekulé's graph regarded as a representation of the covariant to the type [9 : 4, 6 : 6], which is the same thing (except that the hydrogen atoms are suppressed) as the graph to the invariant [15 : 4, 6 ; 1, 6 : 0], would serve to show it to be illusory as previously inferred from other considerations.

ERRATA IN THE PART OF THIS MEMOIR INCLUDED IN THE PRECEDING NUMBER OF THE JOURNAL, pp. 64–104.

Page 66, line 4, *for* quartic *read* quadric.
Page 66, line 4, (from foot) *for* irrefragible *read* irrefragable.
Page 70, line 17, *after* covariant *insert* of.
Page 74, line 6, *for* the second *read* Fig. 15.
Page 74, lines 25, 29, *for* Fig. 43 *read* Fig. 45.
Page 76, line 11, *for* pluralty *read* plurality.
Page 77, line 5, (from foot), *for* CA *read* CB.
Page 79, last paragraph, *for* x *read* y.
Page 80, line 10, *for* a_2d *read* a^2d.
Page 86, line 26, *for* enparametric *read* henparametric.
Page 87, line 12, *for* expressions *read* expression.
Page 91, line 15, *for* j *read* i.
Page 93, line 12, *for* abe *read* ace.
Page 93, line 16, *for* multiplier *read* multiple.
Page 94, line 2, *for* $-\frac{1}{16}(\alpha\beta$ *read* $-\frac{1}{216}(\alpha\beta$.
Page 97, line 5, (from foot,) *for* a $\lambda:\mu:\nu$ *read* as $\lambda:\mu:\nu$.
Page 98, line 9, *for* $I^2U_2=a'U_1+b'U_2+c'U_3$ *read* I^2U_1. $=aU_1+bU_2+cU_3$.
Page 98, line 18, *for* which makes *read* which make.
Page 99, line 5, *for* none *read* no.
Page 99, line 19, *for* $\lambda U+\mu V$ *read* $\Lambda U+\mathrm{M}V$.
Page 101, line 6, *dele period at end of line.*
Page 101, line 20, *after* biologists *insert* term.
Page 101, line 3, (from foot,) *for* y_r *read* y^r.
Page 102, middle of page, *all the indices of* ϵ *should be subscript.*
Page 102, foot-note, *for* $q=1$ *read* $q=i$.
Page 104, last line of text, *for* conclusiou *read* conclusion.

EXTRACT OF A LETTER TO MR. SYLVESTER FROM PROF. CLIFFORD OF UNIVERSITY COLLEGE, LONDON.

THE subjoined matter is so exceedingly interesting and throws such a flood of light on the chemico-algebraical theory, that I have been unable to resist the temptation to insert it in the *Journal*, without waiting to obtain the writer's permission to do so, for which there is not time available between the date of its receipt and my proximate departure for Europe. It is written from Gibraltar, whither Professor Clifford has been ordered to recruit his health, a treasure which he ought to feel bound to guard as a sacred trust for the benefit of the whole mathematical world. J. J. S.

"The new Journal I look forward to with the greatest interest: it will be the only English periodical in which one will have room to print formulæ, except the Philosophical Transactions. I had designed for you a series of papers on the application of Grassmann's methods, but there is only one of them fit for printing yet. It is an *explanation* of the laws of quaternions and of my biquaternions by resolving the units into factors having simpler laws of multiplication; a determination of the corresponding systems for space of any number of dimensions; and a proof that the resulting algebra is a *compound* (in Peirce's sense) of quaternion algebras. It thus appears that quaternions are the last word of geometry in regard to complex algebras.

"Another of them was to be about the very thing you speak of, which was communicated to the British Association at Bristol, *not* Bradford. There is no question of reclamation, because the whole thing is really no more than a translation into other language of your own theories published ages ago in the Cambridge Mathematical Journal.* I have a strong impression that you

* Not having access to the reports of the British Association, I wrote to Professor Clifford to the effect that I had an impression that he had made a communication to the meeting at Bradford which had some affinity to my chemico-algebraical theory and requesting him, in that case, to claim back what was his own.

I think he overstates the obligations which he alleges to my previous papers. At all events he has more than reconquered his title to the merit of the first conception by the completeness he has imparted to it, for whilst I was only able, in certain cases, to represent in terms of the roots of the parent quantic, the quantitative constitution of a form pictured by a graph, through the instrumentality of the reciprocal figure, he has given a direct rule for finding in all cases the algebraical content of a graph in terms of the coefficients, from the graph itself. In a word he has found the universal pass key to the *quantification of graphs*. It seems to me in the highest degree improbable that our joint speculations should not eventually find their embodiment in chemical doctrine proper, and I think that young chemists desirous to raise their science to its proper rank would act wisely in making themselves masters betimes, of the theory of algebraical forms. What mechanics is to physics, that I think algebraical morphology, founded at option on the theory of partitions or ideal elements, or both, is destined to be to the chemistry of the future. Brodie, in his Chemical Calculus, seems to have had an instinctive appreciation of this truth. I have previously called attention to the fact that invariants and isomerism are sister theories.

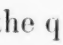
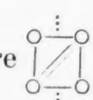
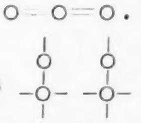
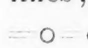
will find there the analogy of covariants and invariants to compound radicals and saturated molecules.*

"I consider forms which are linear in a certain number of sets of k variables each. To fix the ideas, suppose $k = 2$ and that I have altogether 6 sets of 2 variables each, namely

$$x_1x_2\,,\quad y_1y_2\,,\quad z_1z_2\,,\quad u_1u_2\,;\quad v_1v_2\,,\quad w_1w_2\,.$$

Suppose the forms are

$$(xyzu)\,,\quad (yzvw)\,,\quad (xv)\,,\quad (uw)\,;$$

viz. $(xyzu)$ means an expression separately linear and homogeneous in the x, the y, the z, and the u, and so for the rest. I observe that in these four forms each set of variables occurs twice. This being so, there is one invariant of the four forms, which is invariant in regard to *independent* transformations of the six sets of variables. This you knew thirty years ago. All I add is: † *to obtain this invariant*, regard the *variables as alternate numbers*, and *simply multiply all the forms together*. By *alternate* numbers ‡ I mean those whose multiplication is polar ($xy = -yx$) and whose squares are zero. The product of the forms will then be equal to the invariant in question multiplied by the product of all the variables. The quartic forms may be represented by the symbol , the quadratics by . Thus the invariant $(xyzu)(yzvw)(xv)(uw)$ will be represented by the figure ; whereas, $(xyzu)$ $(yzvw)(xu)(vw)$ is this form . The former is clearly the product of the two quartic covariants  got by cutting it across the dotted lines; while the latter is the product of the quadricovariants , . A *bond* between two forms means a set of variables common to

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* I feel induced to say of Professor Clifford what a great master in another field is reported to have done of a certain famous "Oxford graduate," "That young man sees many things in my works which I was not before aware was contained in them," and to regard his generous confessions of obligation as the graceful homage of a young athlete to a half-spent veteran in the arena. What I have seen as in a glass darkly by the uncertain light of analogy, he has viewed by direct vision and brought out into the full blaze of day.

† "All that Professor Clifford adds" is the very pith and marrow of the matter which before was wanting.

‡ The term *alternate numbers* (Hankel's, if my memory serves me right) applied to letters, subject (like Hamilton's i, j, k,) to the law of polar multiplication, seems to me very inadvisable to introduce into the subject. There is no question of quantity, still less of number; and the epithet *alternate* has no meaning except by oblique reference to alternants in the sense of determinants. I think the right term is that first employed by myself and which I shall certainly continue to employ undeviatingly, viz. "polar elements." In like manner I protest against my most expressive and suggestive word "cumulants" being ignored by Mr. Muir and replaced by the unmeaning and ill chosen word "continuants." J. J. S.

them. Of course, we may regard two or more of the forms as identical, and so form invariants of a single form; thus  is the discriminant of a cubic.* . . . . . . Of course, the main thing is to pass from this system of separate variables to that in which the same variables occur to higher orders in the same form, or back again — what you call 'unravelment.' . . . . . .

"The part of the theory which astonished me most is its application to *intergredient* variables when the number in a set is greater than 3, — such as the six co-ordinates of a line in the case of quaternary forms. When the original variables are regarded as alternate numbers, these intergredients are simply their binary products. Thus by simply multiplying the linear forms representing two planes, we get an intergredient form representing their line of intersection. And so generally, whatever be the number of variables in a set, the intergredient variables are merely their products so many together. With this understanding, the product of a set of forms in which the variables are regarded as alternate numbers is the *only* invariant or covariant of the forms which possess certain definite characters of invariance.

"The ordinary theory of symmetrical forms seems to me to bear the same relation to this one (of forms linear in several sets of variables) that a boulder does to a crystal — all the angles rounded off so that you can't see through it so clearly. . . . . ."

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* I will take the example of this figure to illustrate Professor Clifford's rule for finding the *algebraical content* of a graph. Let the bonds be called $\begin{smallmatrix} x & & t & \\ & z & & v. \\ y & & u & \end{smallmatrix}$ Then there will be 4 forms corresponding to the 4 apices or atoms, viz.

$$\begin{gathered}(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)(x_1, x_2)(y_1, y_2)(z_1, z_2),\\ (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)(z_1, z_2)(t_1, t_2)(u_1, u_2),\\ (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)(t_1, t_2)(u_1, u_2)(v_1, v_2),\\ (d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8)(v_1, v_2)(x_1, x_2)(y_1, y_2),\end{gathered}$$

where all the x, y, z, t, u, v letters are to be regarded as *polar elements*. Take the polar product of these forms; the coefficient of

$$x_1 . x_2 . y_1 . y_2 . z_1 . z_2 . t_1 . t_2 . u_1 . u_2 . v_1 . v_2$$

will be an invariant of 3 lineo-lineo-linear forms.

If we make the values identical for the same index, whatever the letter which it affects, it becomes an invariant of a single lineo-lineo-linear form; and finally, if we make the coefficients of $x_1y_1z_2$, $y_1z_1x_2$, $z_1x_1y_2$ all alike, and again the coefficients of $x_2y_2z_1$, $y_2z_2x_1$, $z_2x_2y_1$ all alike, and identify the letters x, y, z, the form becomes a binary cubic and the invariant becomes its discriminant. [*Quaere*, whether this beautiful use of the method of polar multiplication is not, in its ultimate essence, identical with Professor Cayley's original method of hyperdeterminants.] We know *à priori* by my permutation-sum test that the algebraical content above indicated will not vanish because $\Sigma (a-b)^2 (a-d)^2 (a-c)(b-d)$ is not zero, whereas the algebraical content of the figure formed by turning round one of each pair of the doubled lines into the position of the two diagonals respectively *will* vanish because the permutation-sum of $\Sigma (a-b)(b-c)(c-d)(d-a)(a-c)(b-d)$ is zero.

J. J. S.

RESEARCHES IN THE LUNAR THEORY.

By G. W. Hill, *Nyack Turnpike, N. Y.*

(Continued from p. 26.)

Chapter II.

Determination of the inequalities which depend only on the ratio of the mean motions of the sun and moon.

If the path of a body, whose motion satisfies the differential equations

$$\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} + \left[\frac{\mu}{r^3} - 3n'^2\right]x = 0,$$
$$\frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} + \frac{\mu}{r^3}y = 0,$$

intersect the axis of x at right angles, the circumstances of motion, before and after the intersection, are identical, but in reverse order with respect to the time. That is, if t be counted from the epoch when the body is on the axis of x, we shall have

$$x = \text{function}\,(t^2)\,, \qquad y = t\,.\,\text{function}\,(t^2)\,.$$

For if, in the differential equations, the signs of y and t are reversed, but that of x left unchanged, the equations are the same as at first.

A similar thing is true if the path intersect the axis of y at right angles; for if the signs of x and t are reversed, while that of y is not altered, the equations undergo no change.

Now it is evident that the body may start from a given point on, and at right angles to, the axis of x, with different velocities; and that, within certain limits, it may reach the axis of y, and cross the same at correspondingly different angles. If the right angle lie between some of these, we judge, from the principle of continuity, that there is some intermediate velocity with which the body would arrive at and cross the axis of y at right angles.

The difficulty of this question does not permit its being treated by a literal analysis; but the tracing of the path of the body, in numerous special cases, by the application of mechanical quadratures to the differential equations, enables us to state the following circumstances: —

If the body be projected at right angles to, and from a point on, the the axis of x, whose distance from the origin is less than $0.33 \ldots \sqrt[3]{\frac{\mu}{n'^2}}$, there is at least one (near the limit there are two) value of the initial velocity, with which the body, in arriving at the axis of y, will cross it at right angles. Beyond this limit it appears no initial velocity will serve to make the body reach the axis of y under the stated condition.

If the body move from one axis to the other and cross both of them perpendicularly, it is plain, from the preceding developments, that its orbit will be a closed curve symmetrical with respect to both axes. Thus is obtained a particular solution of the differential equations. While the general integrals involve four arbitrary constants, this solution, it is plain, has but two, which may be taken to be the distance from the origin at which the body crosses the axis of x and the time of crossing.

Certain considerations, connected with the employment of Fourier's Theorem and the possibility of developing functions in infinite series of periodic terms, show that, in this solution, the co-ordinates of the body can be represented, in a convergent manner, by series of the following form,

$$x = A_0 \cos [\nu (t - t_0)] + A_1 \cos 3 [\nu (t - t_0)] + A_2 \cos 5 [\nu (t - t_0)] + \ldots,$$

$$y = B_0 \sin [\nu (t - t_0)] + B_1 \sin 3 [\nu (t - t_0)] + B_2 \sin 5 [\nu (t - t_0)] + \ldots,$$

where t_0 denotes the time the body crosses the axis of x, and $\frac{2\pi}{\nu}$ is the time of a complete revolution of the body about the origin. We may regard ν and t_0 as the arbitrary constants introduced by integration; the coefficients $A_0, A_1 \ldots B_1, B_0 \ldots$ are functions of μ, n' and ν.

For convenience sake we may put

$$A_i = \mathrm{a}_i + \mathrm{a}_{-i-1}, \qquad B_i = \mathrm{a}_i - \mathrm{a}_{-i-1}.$$

Then, τ being put for $\nu (t - t_0)$, the series, given above, may be written

$$x = \Sigma_i . \mathrm{a}_i \cos (2i + 1) \tau,$$

$$y = \Sigma_i . \mathrm{a}_i \sin (2i + 1) \tau,$$

the summation being extended to all integral values positive and negative, zero included, for i. By adopting polar co-ordinates such that

$$x = r \cos \phi, \qquad y = r \sin \phi,$$

and writing v for $\phi - \tau$, that is, for the excess of the true over the mean longitude of the moon, the last equations are equivalent to

$$r \cos v = \Sigma_i . \mathrm{a}_i \cos 2i\tau,$$
$$r \sin v = \Sigma_i . \mathrm{a}_i \sin 2i\tau.$$

In order to avoid the multiplication of series of sines and cosines, and reduce everything to an algebraic form, for x and y, we substitute the imaginary variables u and s, and put $\zeta = \varepsilon^{\tau\sqrt{-1}}$. We have then

$$u = \Sigma_i . \mathrm{a}_i \zeta^{2i+1}, \qquad s = \Sigma_i . \mathrm{a}_{-i-1} \zeta^{2i+1}.$$

ζ will always be employed as the independent variable in place of t or τ. Denoting the operation $\zeta \frac{d}{d\zeta} = -\sqrt{-1} \frac{d}{d\tau}$ by the symbol D, so that, in general,

$$D(a\zeta^i) = ia\zeta^i,$$

and taking the liberty of separating this symbol as if it were a multiplier, and moreover putting

$$\mathrm{m} = \frac{n'}{\nu} = \frac{n'}{n - n'}, \qquad \varkappa = \frac{\mu}{\nu^2},$$

the differential equations, determining u and s, given in the preceding chapter, may be written

$$\left[D^2 + 2\mathrm{m}D + \frac{3}{2}\mathrm{m}^2 - \frac{\varkappa}{(us)^{\frac{3}{2}}}\right] u + \frac{3}{2}\mathrm{m}^2 s = 0,$$
$$\left[D^2 - 2\mathrm{m}D + \frac{3}{2}\mathrm{m}^2 - \frac{\varkappa}{(us)^{\frac{3}{2}}}\right] s + \frac{3}{2}\mathrm{m}^2 u = 0.$$

It will be noticed that either of these equations can be derived from the other by interchanging u and s and reversing the sign of m or D. We may also remind the reader that they determine rigorously all the parts of the lunar coordinates which depend only on the ratio of the mean motions of the sun and moon and on the lunar eccentricity. The Jacobian integral, in the present notation, is

$$Du \,.\, Ds + \frac{2\varkappa}{(us)^{\frac{1}{2}}} + \frac{3}{4}\mathrm{m}^2 (u + s)^2 = C.$$

The most ready method of getting the values of the coefficients a_i, is that of undetermined coefficients; the values of u and s, expressed by the preceding summations with reference to i, being substituted in the differential equations, the resulting coefficient of each power of ζ, in the left members, is equated to zero, which furnishes a series of equations of condition sufficient

to determine all the quantities a_i. For this purpose we may evidently employ any two independent combinations of the three equations last written, and it will be advisable to form these combinations in such a manner that the process of deriving the equations of condition may be facilitated in the largest degree. Now it will be recognized that the presence of the term $\frac{\varkappa}{(us)^{\frac{3}{2}}}$, in one of the factors of the differential equations, is a hindrance to their ready integration, being the single thing which prevents them from being linear with constant coefficients. Hence we avail ourselves of the possibility of eliminating it. Multiplying the first differential equation by s, and the second by u, and taking, in succession, the sum and difference,

$$uD^2s + sD^2u - 2\mathrm{m}\,(uDs - sDu) - \frac{2\varkappa}{(us)^{\frac{1}{2}}} + \frac{3}{2}\,\mathrm{m}^2\,(u+s)^2 = 0,$$

$$uD^2s - sD^2u - 2\mathrm{m}\,(uDs + sDu) \qquad\qquad + \frac{3}{2}\,\mathrm{m}^2\,(u^2 - s^2) = 0,$$

then, adding to the first of these the integral equation, and retaining the second as it is, we have, as the final differential equations to be employed,

$$D^2\,(us) - Du\,.\,Ds - 2\mathrm{m}\,(uDs - sDu) + \frac{9}{4}\,\mathrm{m}^2\,(u+s)^2 = C,$$

$$D\,(uDs - sDu - 2\mathrm{m}us) + \frac{3}{2}\,\mathrm{m}^2\,(u^2 - s^2) \qquad\qquad = 0.$$

It must be pointed out, however, that these equations are not, in all respects, a complete substitute for the original equations. It will be seen that μ or $\varkappa$, an essential element in the problem, has disappeared from them, and that, in integration, an arbitrary constant, in excess of those admissible, will present itself. This will be eliminated by substituting the integrals found in one of the original differential equations, in which μ or $\varkappa$ is present; the result being an equation of condition by which the superfluous constant can be expressed in terms of μ and the remaining constants.

We remark that the left members of our differential equations are homogeneous and of two dimensions with respect to u and s. If the first were differentiated, the constant C would disappear, and both equations would be homogeneous in all their terms. This property renders them exceedingly useful when equations of condition are to be obtained between the coefficients of the different periodic terms of the lunar co-ordinates, and it is for this purpose that we have given them their present form.

From the signification of the symbol D,

$$Du = \Sigma_i\,.\,(2i+1)\,\mathrm{a}_i\zeta^{2i+1}, \qquad Ds = \Sigma_i\,.\,(2i+1)\,\mathrm{a}_{-i-1}\zeta^{2i+1},$$
$$D^2u = \Sigma_i\,.\,(2i+1)^2\,\mathrm{a}_i\zeta^{2i+1}, \qquad D^2s = \Sigma_i\,.\,(2i+1)^2\,\mathrm{a}_{-i-1}\zeta^{2i+1};$$

also

$$us = \Sigma_j\,.\,[\Sigma_i\,.\,\mathrm{a}_i\mathrm{a}_{i-j}]\,\zeta^{2j},$$
$$u^2 = \Sigma_j\,.\,[\Sigma_i\,.\,\mathrm{a}_i\mathrm{a}_{-i+j-1}]\,\zeta^{2j},$$
$$s^2 = \Sigma_j\,.\,[\Sigma_i\,.\,\mathrm{a}_i\mathrm{a}_{-i-j-1}]\,\zeta^{2j},$$
$$Du\,.\,Ds = -\Sigma_j\,.\,[\Sigma_i\,.\,(2i+1)(2i-2j+1)\,\mathrm{a}_i\mathrm{a}_{i-j}]\,\zeta^{2j},$$
$$uDs - sDu = -2\Sigma_j\,.\,[\Sigma_i\,.\,(2i-j+1)\,\mathrm{a}_i\mathrm{a}_{i-j}]\,\zeta^{2j},$$

where the summations with reference to j have the same extension as those with reference to i. On substituting these expressions in the differential equations, and equating the general coefficients of ζ^{2j} to zero, we get

$$\Sigma_i.\left[(2i+1)(2i-2j+1) + 4j^2 + 4\,(2i-j+1)\,\mathrm{m} + \frac{9}{2}\,\mathrm{m}^2\right]\mathrm{a}_i\mathrm{a}_{i-j}$$
$$+\frac{9}{4}\,\mathrm{m}^2\,\Sigma_i.\left[\mathrm{a}_i\mathrm{a}_{-i+j-1} + \mathrm{a}_i\mathrm{a}_{-i-j-1}\right] = 0,$$

$$4j\,\Sigma_i\,.\left[2i - j + 1 + \mathrm{m}\right]\mathrm{a}_i\mathrm{a}_{i-j} - \frac{3}{2}\,\mathrm{m}^2\,\Sigma_i.\left[\mathrm{a}_i\mathrm{a}_{-i+j-1} - \mathrm{a}_i\mathrm{a}_{-i-j-1}\right] = 0,$$

which hold for all integral values of j both positive and negative except that, when $j = 0$, the right member of the first equation is C instead of 0; but as the second equation is an identity for $j = 0$, for the present this value of j will be excluded from consideration.

By multiplying the first equation by 2, and the second by 3, and taking in succession the difference and sum, the simpler forms are obtained,

$$\Sigma_i.\,[8i^2 - 8\,(4j-1)\,i + 20j^2 - 16j + 2 + 4\,(4i - 5j + 2)\,\mathrm{m} + 9\mathrm{m}^2]\,\mathrm{a}_i\mathrm{a}_{i-j}$$
$$+\,9\mathrm{m}^2\,\Sigma_i.\mathrm{a}_i\mathrm{a}_{-i+j-1} = 0,$$
$$\Sigma_i.\,[8i^2 + 8\,(2j+1)\,i - 4j^2 + 8j + 2 + 4\,(4i + j + 2)\,\mathrm{m} + 9\mathrm{m}^2]\,\mathrm{a}_i\mathrm{a}_{i-j}$$
$$+\,9\mathrm{m}^2\,\Sigma_i.\,\mathrm{a}_i\mathrm{a}_{-i-j-1} = 0.$$

These two equations are not distinct from each other, when negative, as well as positive values, are attributed to j. For if, in the expression under the first sign of summation in the first equation, we substitute, which is allowable, for i, $i-j$, and $-j$ for j throughout the equation, the result is identical with the second equation. This is explained by the fact that we get all the independent equations of condition, these equations are capable of furnishing, by attributing only positive values to j. Hence, allowing j to receive positive

and negative values, all the equations of condition can be represented by a unique formula.

Although the number of these equations is infinite, and also that of the coefficients a_i, it is not difficult to see that the first ought to be regarded as one less than the second; and that, in consequence of the bi-dimensional character of the equations, they suffice to determine the ratio of any two of the quantities a_i in terms of m. It will be seen, from developments to be given shortly, that if m is regarded as a small quantity of the first order, a_i is of the $\pm 2i^{th}$ order. It will be advisable then to select a_0 as the coefficient to which to refer all the rest; and we shall have, in general,

$$a_i = a_0 F(\mathrm{m}) .$$

The equations of condition, as written above, determine the a_i in pairs; that is, if we put $j = 1$, we have the equations suitable for determining a_1 and a_{-1}, and, in general, the equations, as written, determine a_j and a_{-j}. And, as they involve both these quantities, it will be advantageous to eliminate approximately each in succession, as far as that can be done without depriving the equations of their bi-dimensional character.

By putting, in succession, in the terms under the first sign of summation, $i = 0$ and $i = j$, it will be found that these equations contain, severally, the terms

$$\begin{aligned}&[20j^2 - 16j + 2 - 4(5j - 2)\,\mathrm{m} + 9\mathrm{m}^2]\, a_0 a_{-j} \\ &\qquad + [-4j^2 - 8j + 2 - 4(j - 2)\,\mathrm{m} + 9\mathrm{m}^2]\, a_0 a_j\,, \\ &[-4j^2 + 8j + 2 + 4(j + 2)\,\mathrm{m} + 9\mathrm{m}^2]\, a_0 a_{-j} \\ &\qquad + [20j^2 + 16j + 2 + 4(5j + 2)\,\mathrm{m} + 9\mathrm{m}^2]\, a_0 a_j\,,\end{aligned}$$

which are the terms of principal moment in determining a_{-j} and a_j. Let us then multiply the first equation by

$$-4j^2 + 8j + 2 + 4(j + 2)\,\mathrm{m} + 9\mathrm{m}^2 ,$$

and the second by

$$-20j^2 + 16j - 2 + 4(5j - 2)\,\mathrm{m} - 9\mathrm{m}^2 ,$$

and, adding the products, divide the whole by

$$48j^2\,[2(4j^2 - 1) - 4\mathrm{m} + \mathrm{m}^2] .$$

Then, adopting the notation

$$[j, i] = -\frac{i}{j}\,\frac{4(j - 1)\,i + 4j^2 + 4j - 2 - 4(i - j + 1)\,\mathrm{m} + \mathrm{m}^2}{2(4j^2 - 1) - 4\mathrm{m} + \mathrm{m}^2} ,$$

$$[j] = -\frac{3\mathrm{m}^2}{16j^2}\,\frac{4j^2 - 8j - 2 - 4(j + 2)\,\mathrm{m} - 9\mathrm{m}^2}{2(4j^2 - 1) - 4\mathrm{m} + \mathrm{m}^2} ,$$

$$(j) = -\frac{3m^2}{16j^2}\frac{20j^2 - 16j + 2 - 4(5j-2)m + 9m^2}{2(4j^2-1) - 4m + m^2},$$

the system of equations, which determines the coefficients a_i, is represented by the unique formula

$$\Sigma_i \cdot \left[[j, i]\, a_i a_{i-j} + [j]\, a_i a_{-i+j-1} + (j)\, a_i a_{-i-j-1}\right] = 0,$$

where j must receive negative as well as positive values. It will be perceived that

$$[j, 0] = 0, \qquad [j, j] = -1;$$

hence the last equation is in a form suitable for determining the value of a_j. The quantities $[j, i]$, $[j]$ and (j) admit of being expressed in a simpler manner; thus

$$[j, i] = -\frac{i}{j} + \frac{4i(j-i)}{j}\frac{j-1-m}{2(4j^2-1) - 4m + m^2},$$

whence

$$[j, i] + [-j, -i] = -\frac{2i}{j} + \frac{8i(j-i)}{2(4j^2-1) - 4m + m^2},$$

$$[j, i] - [-j, -i] = \frac{8i(i-j)}{j}\frac{1+m}{2(4j^2-1) - 4m + m^2};$$

in addition

$$[j] = \frac{27}{16j^2}m^2 - \frac{3}{4j^2}\frac{19j^2 - 2j - 5 - (j+11)m}{2(4j^2-1) - 4m + m^2}m^2,$$

$$(j) = -\frac{27}{16j^2}m^2 + \frac{3}{4j^2}\frac{13j^2 + 4j - 5 + (5j-11)m}{2(4j^2-1) - 4m + m^2}m^2,$$

$$[j] + (-j) = -\frac{3}{2j}\frac{3j + 1 + 2m}{2(4j^2-1) - 4m + m^2}m^2,$$

$$[j] - (-j) = \frac{27}{8j^2}m^2 - \frac{3}{2j^2}\frac{16j^2 - 3j - 5 - (3j+11)m}{2(4j^2-1) - 4m + m^2}m^2.$$

In making a first approximation to the values of the coefficients, one of the terms of the equation may be omitted; for, when j is positive, the term $\Sigma_i \cdot (j)\, a_i a_{-i-j-1}$ is a quantity four orders higher than that of the terms of the lowest order contained in the equation; and, when j is negative, the same thing is true of $\Sigma_i \cdot [j]\, a_i a_{-i+j-1}$. Hence, with this limitation, the equation may be written in the two forms

$$\Sigma_i \cdot \left[[j, i]\, a_i a_{i-j} + [j]\, a_i a_{-i+j-1}\right] = 0,$$

$$\Sigma_i \cdot \left[[-j, i]\, a_i a_{i+j} + (-j)\, a_i a_{-i+j-1}\right] = 0.$$

where j takes only positive values.

From these two equations, by omitting all terms but those of the lowest order, we derive the following series of equations, determining the coefficients to the first degree of approximation,

$$
\begin{aligned}
a_0a_1 &= [1]\, a_0a_0\,, \\
a_0a_{-1} &= (-1)\, a_0a_0\,, \\
a_0a_2 &= [2]\, [a_0a_1 + a_1a_0] + [2,\, 1]\, a_1a_{-1}\,, \\
a_0a_{-2} &= (-2)\, [a_0a_1 + a_1a_0] + [-2, -1]\, a_1a_{-1}\,, \\
a_0a_3 &= [3]\, [a_0a_2 + a_1a_1 + a_2a_0] + [3,\, 1]\, a_1a_{-2} + [3,\, 2]\, a_2a_{-1}\,, \\
a_0a_{-3} &= (-3) [a_0a_2 + a_1a_1 + a_2a_0] + [-3, -1]\, a_{-1}a_2 + [-3, -2]\, a_{-2}a_1\,, \\
a_0a_4 &= [4]\, [a_0a_3 + a_1a_2 + a_2a_1 + a_3a_0] + [4,\, 1]\, a_1a_{-3} + [4,\, 2]\, a_2a_{-2} + [4,\, 3]\, a_3a_{-1}\,, \\
a_0a_{-4} &= (-4) [a_0a_3 + a_1a_2 + a_2a_1 + a_3a_0] + [-4, -1]\, a_{-1}a_3 + [-4, -2]\, a_{-2}a_2 \\
&\quad + [-4, -3]\, a_{-3}a_1\,, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

The law of these equations is quite apparent, and they can easily be extended as far as desired. The first two give the values of a_1 and a_{-1}, the following two the values of a_2 and a_{-2} by means of the values of a_1 and a_{-1} already obtained, and so on, every two equations of the series giving the values of two coefficients by means of the values of all those which precede in the order of enumeration. A glance at the composition of these equations must convince us that all attempts to write explicitly, even this approximate value of a_i, would be unsuccessful on account of the excessive multiplicity of the terms. However, they may be regarded, in some sense, as giving the law of this approximate solution, since they exhibit clearly the mode in which each coefficient depends on all those which precede it. As to the degree of approximation afforded by these equations, when the values are expanded in series of ascending powers of m, the first four terms are obtained correctly in the case of each coefficient. Thus a_1 and a_{-1} are affected with errors of the 6th order, a_2 and a_{-2} with errors of the 8th order, a_3 and a_{-3} with errors of the 10th order, and so on.

The values of these quantities can be determined either in the literal form, where the parameter m is left indeterminate, as has been done by Plana and Delaunay, or as numbers, which mode has been followed by all the earlier lunar-theorists and Hansen. In the latter case, one will begin by computing the numerical values of the quantities $[j,\, i]$, $[j]$ and (j), corresponding to the assumed value of m, for all necessary values of the integers i and j.

The great advantage of our equations consists in this, that we are able to extend the approximation as far as we wish, simply by writing explicitly the terms, our symbols giving the law of the coefficients. How rapid is the approximation in the terms of these equations will be apparent, when we say, that, after a certain number of terms are written, in order to carry this four orders higher, it is necessary to add to each of them only four new terms; and, thereafter, every addition of four terms enables us to carry the approximation four orders farther.

The process which may be followed to obtain the values of the a_i with any desired degree of accuracy, is this: — the first approximate values will be got from the preceding group of equations until the a_i become of orders intended to be neglected; then one will recommence at the beginning, using the equations each augmented by the terms necessary to carry the approximation four orders higher; substituting in the new terms the values obtained from the first approximation, and, in the old, ascertaining what changes are produced by employing the more exact values instead of the first approximations. A second return to the beginning of the work will, in like manner, push the degree of exactitude four orders higher. In this way any required degree of approximation may be attained.

Whatever advantage the present process may have over those previously employed is plainly due to the use of the indeterminate integers i and j, which, although much used in the planetary theories, no one seems to have thought of introducing into the lunar theory. This enables us to perform a large mass of operations once for all.

For the purpose of making evident the preceding assertions, and because we shall have occasion to use them, we write below the equations determining the coefficients a_i correct to quantities of the 13th order inclusive.

$$
\begin{aligned}
a_0a_1 &= [1]\,[a_0^2 + 2a_{-1}a_1 + 2a_{-2}a_2] + (1)\,[a_{-1}^2 + 2a_0a_{-2} + 2a_1a_{-3}] \\
&\quad + [1,-2]\,a_{-2}a_{-3} + [1,-1]\,a_{-1}a_{-2} + [1,2]\,a_2a_1 + [1,3]\,a_3a_2\,, \\
a_0a_{-1} &= [-1]\,[a_{-1}^2 + 2a_0a_{-2} + 2a_1a_{-3}] + (-1)\,[a_0^2 + 2a_{-1}a_1 + 2a_{-2}a_2] \\
&\quad + [-1,-3]\,a_{-3}a_{-2} + [-1,-2]\,a_{-2}a_{-1} + [-1,1]\,a_1a_2 + [-1,2]\,a_2a_3\,, \\
a_0a_2 &= [2]\,[2a_0a_1 + 2a_{-1}a_2 + 2a_{-2}a_3] + (2)\,[2a_{-1}a_{-2} + 2a_0a_{-3} + 2a_1a_{-4}] \\
&\quad + [2,-2]\,a_{-2}a_{-4} + [2,-1]\,a_{-1}a_{-3} + [2,1]\,a_1a_{-1} + [2,3]\,a_3a_1 + [2,4]\,a_4a_2\,, \\
a_0a_{-2} &= [-2]\,[2a_{-1}a_{-2} + 2a_0a_{-3} + 2a_1a_{-4}] + (-2)\,[2a_0a_1 + 2a_{-1}a_2 + 2a_{-2}a_3] \\
&\quad + [-2,-4]\,a_{-4}a_{-2} + [-2,-3]\,a_{-3}a_{-1} + [-2,-1]\,a_{-1}a_1 + [-2,1]\,a_1a_3 \\
&\quad + [-2,2]\,a_2a_4\,,
\end{aligned}
$$

$$\begin{aligned}
a_0a_3 &= [3]\,[a_1^2 + 2a_0a_2 + 2a_{-1}a_3] + (3)\,[a_{-2}^2 + 2a_{-1}a_{-3} + 2a_0a_{-4}] \\
&\quad + [3, -1]\,a_{-1}a_{-4} + [3, 1]\,a_1a_{-2} + [3, 2]\,a_2a_{-1} + [3, 4]\,a_4a_1\,, \\
a_0a_{-3} &= [-3]\,[a_{-2}^2 + 2a_{-1}a_{-3} + 2a_0a_{-4}] + (-3)\,[a_1^2 + 2a_0a_2 + 2a_{-1}a_3] \\
&\quad + [-3,-4]\,a_{-4}a_{-1} + [-3,-2]\,a_{-2}a_1 + [-3,-1]\,a_{-1}a_2 + [-3, 1]\,a_1a_4\,, \\
a_0a_4 &= [4]\,[2a_1a_2 + 2a_0a_3 + 2a_{-1}a_4] + (4)\,[2a_{-2}a_{-3} + 2a_{-1}a_{-4} + 2a_0a_{-5}] \\
&\quad + [4,-1]\,a_{-1}a_{-5} + [4, 1]\,a_1a_{-3} + [4, 2]\,a_2a_{-2} + [4,3]\,a_3a_{-1} + [4,5]\,a_5a_1\,, \\
a_0a_{-4} &= [-4]\,[2a_{-2}a_{-3} + 2a_{-1}a_{-4} + 2a_0a_{-5}] + (-4)\,[2a_1a_2 + 2a_0a_3 + 2a_{-1}a_4] \\
&\quad + [-4,-5]\,a_{-5}a_{-1} + [-4,-3]\,a_{-3}a_1 + [-4,-2]\,a_{-2}a_2 + [-4,-1]\,a_{-1}a_3 \\
&\quad + [-4, 1]\,a_1a_5\,, \\
a_0a_5 &= [5]\,[a_2^2 + 2a_1a_3 + 2a_0a_4] \\
&\quad + [5, 1]\,a_1a_{-4} + [5, 2]\,a_2a_{-3} + [5, 3]\,a_3a_{-2} + [5, 4]\,a_4a_{-1}\,, \\
a_0a_{-5} &= (-5)\,[a_2^2 + 2a_1a_3 + 2a_0a_4] \\
&\quad + [-5,-4]\,a_{-4}a_1 + [-5,-3]\,a_{-3}a_2 + [-5,-2]\,a_{-2}a_3 + [-5,-1]\,a_{-1}a_4\,, \\
a_0a_6 &= [6]\,[2a_2a_3 + 2a_1a_4 + 2a_0a_5] \\
&\quad + [6, 1]\,a_1a_{-5} + [6,2]\,a_2a_{-4} + [6, 3]\,a_3a_{-3} + [6, 4]\,a_4a_{-2} + [6, 5]\,a_5a_{-1}\,, \\
a_0a_6 &= (-6)\,[2a_2a_3 + 2a_1a_4 + 2a_0a_5] \\
&\quad + [-6,-5]\,a_{-5}a_1 + [-6,-4]\,a_{-4}a_2 + [-6,-3]\,a_{-3}a_3 + [-6,-2]\,a_{-2}a_4 \\
&\quad + [-6, -1]\,a_{-1}a_5\,.
\end{aligned}$$

In the first approximation

$$\begin{aligned}
a_1 &= [1]\,a_0\,, \\
a_{-1} &= (-1)\,a_0\,, \\
a_2 &= [1]\Big[2\,(2) + [2, 1]\,(-1)\Big]\,a_0\,, \\
a_{-2} &= [1]\Big[2\,(-2) + [-2, -1]\,(-1)\Big]\,a_0\,,
\end{aligned}$$

or, explicitly in terms of m,

$$\begin{aligned}
a_1 &= \frac{3}{16}\,\frac{6 + 12m + 9m^2}{6 - 4m + m^2}\,m^2a_0\,, \\
a_{-1} &= -\frac{3}{16}\,\frac{38 + 28m + 9m^2}{6 - 4m + m^2}\,m^2a_0\,,
\end{aligned}$$

and, after some reductions,

$$\begin{aligned}
a_2 &= \frac{27}{256}\,\frac{2 + 4m + 3m^2}{[6 - 4m + m^2]\,[30 - 4m + m^2]}\left[238 + 40m + 9m^2 - 32\,\frac{29 - 35m}{6 - 4m + m^2}\right]m^4a_0\,, \\
a_{-2} &= \frac{27}{64}\,\frac{2 + 4m + 3m^2}{[6 - 4m + m^2]\,[30 - 4m + m^2]}\left[-28 - 7m + 24\,\frac{7 - m}{6 - 4m + m^2}\right]m^4a_0\,.
\end{aligned}$$

It is evident that, however far the approximation may be carried, the only quantities, involved as divisors in the values of the a_i, are the trinomials, whose general expression is

$$2(4j^2-1)-4m+m^2,$$

or, particularizing, the series of divisors is

$$\begin{aligned}&6-4m+m^2,\\&30-4m+m^2,\\&70-4m+m^2,\\&\quad\cdots\cdots\end{aligned}$$

It will be remarked that they differ only in their first terms, which are independent of m. Hence any expression, involving several divisors, can always be separated into several parts, each involving only one divisor, without any actual division by a trinomial in m. For instance,

$$\frac{1}{[6-4m+m^2][30-4m+m^2]}=\frac{1}{24}\frac{1}{6-4m+m^2}-\frac{1}{24}\frac{1}{30-4m+m^2},$$

$$\frac{1}{[6-4m+m^2]^2[30-4m+m^2]}=\frac{1}{24}\frac{1}{[6-4m+m^2]^2}-\frac{1}{24^2}\frac{1}{6-4m+m^2}$$
$$+\frac{1}{24^2}\frac{1}{30-4m+m^2}.$$

Moreover when, after this transformation, any numerator contains more or other powers of m than two consecutive powers, it is clear it may be reduced so as to contain only these by eliminating the higher powers through subtracting certain multiples of the divisor which appears in the denominator, or, in other words, the fraction may be treated as if it were improper.

From this we gather that the value of a_i can be expressed thus

$$\begin{aligned}\frac{a_i}{a_0}=M_0&+\frac{M_1}{6-4m+m^2}+\frac{M_2}{[6-4m+m^2]^2}+\frac{M_3}{[6-4m+m^2]^3}+\cdots\\&+\frac{N_1}{30-4m+m^2}+\frac{N_2}{[30-4m+m^2]^2}+\frac{N_3}{[30-4m+m^2]^3}+\cdots\\&+\frac{P_1}{70-4m+m^2}+\frac{P_2}{[70-4m+m^2]^2}+\frac{P_3}{[70-4m+m^2]^3}+\cdots\\&+\quad\cdots\cdots\cdots\cdots\end{aligned}$$

where $M_0, M_1 \ldots N_1, N_2 \ldots P_1, P_2 \ldots$ are entire functions of m each of the form $$Am^k+Bm^{k+1}.$$
The advantage of this method of treatment consists in that nothing, which is given by the successive approximations, would be lost, as must be the case

when the values are expanded in series of ascending powers of m. The preceding expressions, when put into this form, become

$$\frac{a_1+a_{-1}}{a_0}=-3\,\frac{2+m}{6-4m+m^2}\,m^2\,,$$

$$\frac{a_1-a_{-1}}{a_0}=3\left[\frac{9}{8}-\frac{4-7m}{6-4m+m^2}\right]m^2\,,$$

$$\frac{a_2+a_{-2}}{a_0}=\frac{3}{16}\left[\frac{243}{16}+\frac{323+109m}{6-4m+m^2}-96\,\frac{23-11m}{[6-4m+m^2]^2}-\frac{215-53m}{30-4m+m^2}\right]m^4\,,$$

$$\frac{a_2-a_{-2}}{a_0}=\frac{3}{32}\left[\frac{243}{8}+\frac{175+563m}{6-4m+m^2}-48\,\frac{89-32m}{[6-4m+m^2]^2}+5\,\frac{361-10m}{30-4m+m^2}\right]m^4.$$

The evident objection to this form for the coefficients is that it makes the several terms very large, and of signs such that they nearly neutralize each other, the sum being very much smaller than any of the component terms. However it may be possible to remedy this imperfection by admitting three terms into the numerators, but, in this way, the problem is indeterminate, infinite variety being possible.

It is remarkable that none of our system of divisors can vanish for any real value of m, since the quadratic equations, obtained by equating them to zero, have all imaginary roots. In this they differ from the binomial divisors met with when the integration is effected in approximations arranged according to ascending powers of the disturbing force.

It is well known that the infinite series, obtained from the development, in ascending powers of m, of any fraction whose numerator is an entire function of m, and its denominator any integral power of a divisor of the previously mentioned series, is convergent, provided that m lies between the two square roots of the absolute term of the divisor. Hence any finite expression in m, involving these divisors, can be developed in such a series, provided that the numerical value of this parameter is less than $\sqrt{6}$. The same, however, cannot be asserted when the expression really forms an infinite series, as it is in the equation just given for the value of $\frac{a_i}{a_0}$. Yet, on account of the simplicity with which these quantities can be expressed in this form, a_1 and a_{-1} containing each a single term, with an error of the sixth order only, this limit is worthy of attention.

If the parameter *m*, hitherto employed by the lunar theorists, is taken as the quantity in powers of which to expand the value of a_i, we shall have

$\mathrm{m} = \dfrac{m}{1 - m}$. And, substituting this value, the principal divisor $6 - 4\mathrm{m} + \mathrm{m}$ becomes $6 - 16m + 11m^2$. Thus the limits, between which m must be contained, in order that convergent series may be obtained where this divisor intervenes, are $\pm\sqrt{\dfrac{6}{11}}$. When we consider how little, in the case of our moon, m exceeds m, it will be plain that the series, in terms of m, are likely to be much more convergent than those in terms of m.

If we inquire what function of m, of the form $\dfrac{\mathrm{m}}{1 + \alpha\mathrm{m}}$, the quantity

$$\frac{M}{[6 - 4\mathrm{m} + \mathrm{m}^2]^k}$$

can be expanded in powers of, with the greatest convergency, it is easily found that α should be $-\dfrac{1}{3}$. Then putting

$$\mathrm{m} = \frac{\mathfrak{m}}{1 + \frac{1}{3}\mathfrak{m}},$$

the divisor $6 - 4\mathrm{m} + \mathrm{m}^2$ is changed into

$$6 + \frac{1}{3}\mathfrak{m}^2,$$

and there is introduced the additional divisor $1 + \dfrac{1}{3}\mathfrak{m}$. Here the series will be convergent provided $\mathfrak{m}$ is less than 3. It is true the terms involving the succeeding divisors $30 - 4\mathrm{m} + \mathrm{m}^2$, &c., are not benefited by this change of parameter, but as they play an inferior *rôle* in this matter, I have chosen $\mathfrak{m}$ as the parameter for the developments of the coefficients a_i in series of ascending powers.

To illustrate this matter, we have, in terms of the parameter $\mathfrak{m}$, and with errors of the sixth order,

$$\frac{\mathrm{a}_1 + \mathrm{a}_{-1}}{\mathrm{a}_0} = -\left[\frac{2 + \frac{1}{6}\mathfrak{m}}{1 + \frac{1}{18}\mathfrak{m}^2} - \frac{1}{1 + \frac{1}{3}\mathfrak{m}}\right]\mathfrak{m}^2,$$

$$\frac{\mathrm{a}_1 - \mathrm{a}_{-1}}{\mathrm{a}_0} = \left[\frac{5 + \frac{7}{6}\mathfrak{m}}{1 + \frac{1}{18}\mathfrak{m}^2} - \frac{7}{1 + \frac{1}{3}\mathfrak{m}} + \frac{\frac{27}{8}}{[1 + \frac{1}{3}\mathfrak{m}]^2}\right]\mathfrak{m}^2.$$

Expanding these expressions in powers of $\mathfrak{m}$, we get

$$\frac{\mathrm{a}_1 + \mathrm{a}_{-1}}{\mathrm{a}_0} = -\left[\mathfrak{m}^2 + \frac{1}{2}\mathfrak{m}^3 - \frac{2}{9}\mathfrak{m}^4 + \frac{1}{36}\mathfrak{m}^5 + \ldots . \right],$$

$$\frac{\mathrm{a}_1 - \mathrm{a}_{-1}}{\mathrm{a}_0} = \frac{11}{8}\mathfrak{m}^2 + \frac{5}{4}\mathfrak{m}^3 + \frac{5}{72}\mathfrak{m}^4 - \frac{11}{36}\mathfrak{m}^5 + \ldots .$$

Let these series be compared with those which correspond to them in the lunar theories of Plana or Delaunay, viz:

$$m^2 + \frac{19}{6}m^3 + \frac{131}{18}m^4 + \frac{383}{27}m^5 + \ldots . ,$$

$$\frac{11}{8}m^2 + \frac{59}{12}m^3 + \frac{893}{72}m^4 + \frac{2855}{108}m^5 + \ldots .$$

The superiority of the former, in convergence and simplicity of numerical coefficients, is manifest.

Much more might be said relative to possible modes of developing the coefficients a_i in series, but we content ourselves with giving their values expanded in powers of m, the series being carried to terms of the ninth order inclusive. The denominators of the numerical fractions are written as products of their prime factors, as, in this form, they can be more readily used, the principal labor in performing operations on these series being the reduction of the several fractional coefficients, to be added together, to a common denominator.

$$\frac{\mathrm{a}_1}{\mathrm{a}_0} = \frac{3}{2^4}\mathrm{m}^2 + \frac{1}{2}\mathrm{m}^3 + \frac{7}{2^2.3}\mathrm{m}^4 + \frac{11}{2^2.3^2}\mathrm{m}^5 - \frac{30749}{2^{12}.3^3}\mathrm{m}^6 - \frac{1010521}{2^{11}.3^4.5}\mathrm{m}^7 - \frac{18445871}{2^{10}.3^5.5^2}\mathrm{m}^8$$
$$- \frac{2114557853}{2^{12}.3^6.5^3}\mathrm{m}^9 \ldots$$

$$\frac{\mathrm{a}_{-1}}{\mathrm{a}_0} = -\frac{19}{2^4}\mathrm{m}^2 - \frac{5}{3}\mathrm{m}^3 - \frac{43}{2^2.3^2}\mathrm{m}^4 - \frac{14}{3^3}\mathrm{m}^5 - \frac{7381}{2^{10}.3^4}\mathrm{m}^6 + \frac{3574153}{2^{11}.3^5.5}\mathrm{m}^7 + \frac{55218889}{2^9.3^6.5^2}\mathrm{m}^8$$
$$+ \frac{13620153029}{2^{12}.3^7.5^3}\mathrm{m}^9 \ldots$$

$$\frac{\mathrm{a}_2}{\mathrm{a}_0} = \frac{25}{2^8}\mathrm{m}^4 + \frac{803}{2^7.3.5}\mathrm{m}^5 + \frac{6109}{2^5.3^2.5^2}\mathrm{m}^6 + \frac{897599}{2^8.3^3.5^3}\mathrm{m}^7 + \frac{237203647}{2^{16}.3^2.5^4}\mathrm{m}^8 - \frac{44461407673}{2^{15}.3^4.5^5.7}\mathrm{m}^9 \ldots$$

$$\frac{\mathrm{a}_{-2}}{\mathrm{a}_0} = 0\mathrm{m}^4 + \frac{23}{2^7.5}\mathrm{m}^5 + \frac{299}{2^5.3.5^2}\mathrm{m}^6 + \frac{56339}{2^8.3^2.5^3}\mathrm{m}^7 + \frac{79400351}{2^{16}.3^2.5^4}\mathrm{m}^8 + \frac{8085846833}{2^{14}.3^4.5^5.7}\mathrm{m}^9 \ldots$$

$$\frac{\mathrm{a}_3}{\mathrm{a}_0} = \frac{833}{2^{12}.3}\mathrm{m}^6 + \frac{27943}{2^{11}.5.7}\mathrm{m}^7 + \frac{12275527}{2^{10}.3^2.5^2.7^2}\mathrm{m}^8 + \frac{27409853579}{2^{12}.3^4.5^3.7^3}\mathrm{m}^9 \ldots$$

$$\frac{\mathrm{a}_{-3}}{\mathrm{a}_0} = \frac{1}{2^6.3}\,\mathrm{m}^6 + \frac{71}{2^7.3.5}\,\mathrm{m}^7 + \frac{46951}{2^8.3^2.5^2.7}\,\mathrm{m}^8 + \frac{14086643}{2^7.3^4.5^3.7^2}\,\mathrm{m}^9 \ldots$$

$$\frac{\mathrm{a}_4}{\mathrm{a}_0} = \frac{3537}{2^{16}}\,\mathrm{m}^8 + \frac{111809667}{2^{17}.3^2.5.7^2}\,\mathrm{m}^9 \ldots$$

$$\frac{\mathrm{a}_{-4}}{\mathrm{a}_0} = \frac{23}{2^{11}.3}\,\mathrm{m}^8 + \frac{1576553}{2^{17}.3^2.7^2}\,\mathrm{m}^9 \ldots$$

These values being substituted in the equations

$$r \cos v = \Sigma_i \,.\, \mathrm{a}_i \cos 2i\tau \,,$$
$$r \sin v = \Sigma_i \,.\, \mathrm{a}_i \sin 2i\tau \,,$$

and the parameter changed to $\mathfrak{m}$, we get

$$\begin{aligned} r \cos v = \mathrm{a}_0 \Big\{ 1 + \Big[& - \mathfrak{m}^2 - \tfrac{1}{2}\,\mathfrak{m}^3 + \tfrac{2}{9}\,\mathfrak{m}^4 - \tfrac{1}{36}\,\mathfrak{m}^5 - \tfrac{106411}{331776}\,\mathfrak{m}^6 + \tfrac{427339}{497664}\,\mathfrak{m}^7 \\ & + \tfrac{25239037}{14929920}\,\mathfrak{m}^8 - \tfrac{732931}{37324800}\,\mathfrak{m}^9 \ldots \Big] \cos 2\tau \\ & + \Big[\tfrac{25}{256}\,\mathfrak{m}^4 + \tfrac{311}{960}\,\mathfrak{m}^5 + \tfrac{9349}{28800}\,\mathfrak{m}^6 - \tfrac{5831}{216000}\,\mathfrak{m}^7 \\ & - \tfrac{164645363}{552960000}\,\mathfrak{m}^8 - \tfrac{11321875589}{19353600000}\,\mathfrak{m}^9 \ldots \Big] \cos 4\tau \\ & + \Big[\tfrac{299}{4096}\,\mathfrak{m}^6 + \tfrac{30193}{107520}\,\mathfrak{m}^7 + \tfrac{379549}{1003520}\,\mathfrak{m}^8 + \tfrac{181908179}{1580544000}\,\mathfrak{m}^9 \ldots \Big] \cos 6\tau \\ & + \Big[\tfrac{11347}{196608}\,\mathfrak{m}^8 + \tfrac{2350381}{9031680}\,\mathfrak{m}^9 \ldots \Big] \cos 8\tau + \ldots \Big\} \,, \end{aligned}$$

$$\begin{aligned} r \sin v = \mathrm{a}_0 \Big\{ \Big[& \tfrac{11}{8}\,\mathfrak{m}^2 + \tfrac{5}{4}\,\mathfrak{m}^3 + \tfrac{5}{72}\,\mathfrak{m}^4 - \tfrac{11}{36}\,\mathfrak{m}^5 - \tfrac{101123}{331776}\,\mathfrak{m}^6 - \tfrac{512239}{276480}\,\mathfrak{m}^7 \\ & - \tfrac{269023019}{74649600}\,\mathfrak{m}^8 - \tfrac{151872119}{93312000}\,\mathfrak{m}^9 \ldots\ldots \Big] \sin 2\tau \\ & + \Big[\tfrac{25}{256}\,\mathfrak{m}^4 + \tfrac{121}{480}\,\mathfrak{m}^5 + \tfrac{5623}{28800}\,\mathfrak{m}^6 - \tfrac{17149}{432000}\,\mathfrak{m}^7 \\ & - \tfrac{3500287}{11520000}\,\mathfrak{m}^8 - \tfrac{43885512859}{58060800000}\,\mathfrak{m}^9 \ldots\ldots \Big] \sin 4\tau \\ & + \Big[\tfrac{769}{12288}\,\mathfrak{m}^6 + \tfrac{24481}{107520}\,\mathfrak{m}^7 + \tfrac{4419347}{15052800}\,\mathfrak{m}^8 + \tfrac{398314169}{4741632000}\,\mathfrak{m}^9 \ldots \Big] \sin 6\tau \\ & + \Big[\tfrac{9875}{196608}\,\mathfrak{m}^8 + \tfrac{32608451}{144506880}\,\mathfrak{m}^9 \ldots\ldots \Big] \sin 8\tau + \ldots \Big\} \,. \end{aligned}$$

Our final differential equations are capable of furnishing only the ratios of the coefficients a_i, hence we must have recourse to one of the original

equations if we wish to determine a_0 as a function of n and μ. By substituting the values

$$u = \Sigma_i \, . \, a_i \zeta^{2i+1}, \qquad s = \Sigma_i \, . \, a_{-i-1} \zeta^{2i+1},$$

in the differential equation

$$\left[D^2 + 2\mathrm{m}D + \frac{3}{2}\mathrm{m}^2 - \frac{\varkappa}{(us)^{\frac{3}{2}}}\right] u + \frac{3}{2}\mathrm{m}^2 s = 0,$$

we obtain

$$\frac{\varkappa u}{(us)^{\frac{3}{2}}} = \Sigma_i \, . \left\{ \left[(2i + 1 + \mathrm{m})^2 + \frac{1}{2}\mathrm{m}^2\right] a_i + \frac{3}{2}\mathrm{m}^2 a_{-i-1} \right\} \zeta^{2i+1}.$$

Considering only the term of this, for which $i = 0$, and supposing that the coefficient of ζ in the expansion of $\frac{a_0^2 u}{(us)^{\frac{3}{2}}}$ is denoted by J, we shall have

$$\frac{\varkappa}{a_0^3} J = 1 + 2\mathrm{m} + \frac{3}{2}\mathrm{m}^2 + \frac{3}{2}\mathrm{m}^2 \frac{a_{-1}}{a_0}.$$

For brevity call the right member of this H; then, since

$$\varkappa = \frac{\mu}{(n - n')^2} = \frac{\mu}{n^2}(1 + \mathrm{m})^2,$$

we shall have

$$a_0 = \left[\frac{\mu}{n^2}\right]^{\frac{1}{3}} \left[\frac{J(1 + \mathrm{m})^2}{H}\right]^{\frac{1}{3}}.$$

The value of H is readily obtained from the value of $\frac{a_{-1}}{a_0}$ given above, and J must be found by substituting the values

$$u = \Sigma_i \, . \, a_i \zeta^{2i+1}, \qquad s = \Sigma_i \, . \, a_{-i-1} \zeta^{2i+1}$$

in $\frac{a_0^2 u}{(us)^{\frac{3}{2}}}$, and taking the coefficient of ζ. We get

$$\begin{aligned} J = 1 &+ \left[\frac{a_1 + a_{-1}}{a_0}\right]^2 \left[\frac{3}{4} + \frac{45}{64}\left[\frac{a_1 + a_{-1}}{a_0}\right]^2 + \frac{15}{8}\frac{a_1 a_{-1}}{a_0} - \frac{15}{2}\frac{a_2 + a_{-2}}{a_0}\right] \\ &+ \frac{a_2 + a_{-2}}{a_0}\left[\frac{3}{4}\frac{a_2 + a_{-2}}{a_0} + 6\frac{a_1 a_{-1}}{a_0^2}\right] + 6\frac{a_1 + a_{-1}}{a_0}\frac{a_1 a_2 + a_{-1} a_{-2}}{a_0^2} \\ &+ 3\frac{a_1 a_{-1}}{a_0^2} + 45\frac{a_1^2 a_{-1}^2}{a_0^4} + 3\frac{a_2 a_{-2}}{a_0^2}, \end{aligned}$$

where the terms neglected are, at lowest, of the tenth order with respect to m. And, explicitly in terms of this parameter,

$$J = 1 + \frac{21}{2^8}\mathrm{m}^4 - \frac{31}{2^5}\mathrm{m}^5 - \frac{53}{2^4}\mathrm{m}^6 - \frac{2707}{2^6 . 3^2}\mathrm{m}^7 - \frac{4201213}{2^{16} . 3^3}\mathrm{m}^8 + \frac{14374939}{2^{15} . 3^3 . 5}\mathrm{m}^9 \ldots.$$

By means of which there is obtained

$$\mathrm{a}_0 = \left[\frac{\mu}{n^2}\right]^{\frac{1}{3}} \Big[1 - \tfrac{1}{6}\,\mathrm{m}^2 + \tfrac{1}{3}\,\mathrm{m}^3 + \tfrac{407}{2304}\,\mathrm{m}^4 - \tfrac{67}{288}\,\mathrm{m}^5 - \tfrac{45293}{41472}\,\mathrm{m}^6$$

$$- \tfrac{8761}{6912}\,\mathrm{m}^7 - \tfrac{4967441}{7962624}\,\mathrm{m}^8 + \tfrac{14829273}{39813120}\,\mathrm{m}^9 \ldots \Big],$$

or, in terms of the parameter $\mathfrak{m}$,

$$\mathrm{a}_0 = \left[\frac{\mu}{n^2}\right]^{\frac{1}{3}} \Big[1 - \tfrac{1}{6}\,\mathfrak{m}^2 + \tfrac{4}{9}\,\mathfrak{m}^3 - \tfrac{163}{768}\,\mathfrak{m}^4 - \tfrac{1147}{5184}\,\mathfrak{m}^5 - \tfrac{79859}{124416}\,\mathfrak{m}^6$$

$$+ \tfrac{4811}{10368}\,\mathfrak{m}^7 + \tfrac{9520295}{71663616}\,\mathfrak{m}^8 + \tfrac{139240651}{1074954240}\,\mathfrak{m}^9 \ldots \Big].$$

The quantity $\left[\frac{\mu}{n^2}\right]^{\frac{1}{3}}$ is usually designated a by the lunar-theorists; and, to make this appear as a factor of the expressions for $r\cos v$ and $r\sin v$, it would be necessary to multiply all the coefficients by the second factor of the preceding expression for a_0. It seems simpler however to retain a_0 as the factor of linear magnitude; for the astronomers have preferred to derive the constant of lunar parallax from direct observation of the moon, or, in other words, they have preferred to consider μ as a seventh element of the orbit; with this view of the matter, there is no incongruity in making a_0 everywhere replace μ.

The expression for a_0 can be obtained in several other ways, which lead to more symmetrical formulæ, and which also serve for verification of all the preceding developments. If, in the preceding equation giving the value of $\frac{\varkappa u}{(us)^{\frac{3}{2}}}$ in terms of ζ, we attribute to τ the value 0, or, which is equivalent, make $\zeta = 1$, we shall have $u = s = \Sigma_i \,.\, \mathrm{a}_i$, and, consequently

$$\frac{\varkappa}{[\Sigma_i \,.\, \mathrm{a}_i]^2} = \Sigma_i \,.\, [(2i + 1 + \mathrm{m})^2 + 2\mathrm{m}^2]\, \mathrm{a}_i \,.$$

And thus, mindful of the value of $\varkappa$ given above, we get

$$\mathrm{a}_0 = \left[\frac{\mu}{n^2}\right]^{\frac{1}{3}} \left[\frac{(1 + \mathrm{m})^2}{\Sigma_i \,.\, \left[(2i + 1 + \mathrm{m})^2 + 2\mathrm{m}^2\right]\frac{\mathrm{a}_i}{\mathrm{a}_0} \,.\, \left[\Sigma_i \,.\, \frac{\mathrm{a}_i}{\mathrm{a}_0}\right]^2}\right]^{\frac{1}{3}}.$$

Again the differential equation

$$\frac{d^2 y}{d\tau^2} + 2\mathrm{m}\frac{dx}{d\tau} + \frac{\varkappa}{r^3}\,y = 0$$

gives

$$\frac{\varkappa}{r^3} \cdot y = \Sigma_i \cdot [(2i + 1 + \text{m})^2 - \text{m}^2] \, \text{a}_i \sin (2i + 1) \, \tau \, ,$$

and, attributing to τ the special value $\frac{\pi}{2}$,

$$\frac{\varkappa}{[\Sigma_i \cdot (-1)^i \, \text{a}_i]^2} = \Sigma_i \cdot (-1)^i \, (2i + 1) \, (2i + 1 + \text{m}) \, \text{a}_i \, .$$

Whence

$$\text{a}_0 = \left[\frac{\mu}{n^2}\right]^{\frac{1}{3}} \left[\frac{(1 + \text{m})^2}{\Sigma_i \cdot (-1)^i \, (2i+1) \, (2i+1+\text{m}) \, \frac{\text{a}_i}{\text{a}_0} \cdot \left[\Sigma_i \cdot (-1)^i \, \frac{\text{a}_i}{\text{a}_0}\right]^2}\right]^{\frac{1}{3}} .$$

When $j = 0$ in the first equation of condition for determining the coefficients a_i, we get a formula expressing C in terms of these quantities, viz.,

$$C = \Sigma_i \cdot \left[(2i + 1 + 2\text{m})^2 + \tfrac{1}{2} \, \text{m}^2\right] \text{a}_i^2 + \tfrac{9}{2} \, \text{m}^2 \, \Sigma_i \cdot \text{a}_i \text{a}_{-i-1} \, ,$$

or neglecting terms of the eighth and higher orders,

$$C = \text{a}_0^2 \left[1 + 4\text{m} + \tfrac{9}{2} \, \text{m}^2 + (9 + 12\text{m} + \tfrac{9}{2} \, \text{m}^2) \, \frac{\text{a}_1^2}{\text{a}_0^2} + (1 - 4\text{m} + \tfrac{9}{2} \, \text{m}^2) \, \frac{\text{a}_{-1}^2}{\text{a}_0^2} + 9\text{m}^2 \, \frac{\text{a}_{-1}}{\text{a}_0}\right]$$

$$= \text{a}_0^2 \left[1 + 4\text{m} + \tfrac{9}{2} \, \text{m}^2 - \tfrac{1147}{2^7} \, \text{m}^4 - \tfrac{1399}{2^5.3} \, \text{m}^5 - \tfrac{2047}{2^8} \, \text{m}^6 + \tfrac{3737}{2^4.3^3} \, \text{m}^7\right] .$$

But the C of Chap. I is obtained by multiplying this C by $\frac{1}{2} \, \nu^2 = \frac{1}{2} \, \frac{n^2}{(1 + \text{m})^2}$. Hence, substituting for a_0 its value, we have

$$C = \tfrac{1}{2} \, (\mu n)^{\frac{2}{3}} \left[1 + 2\text{m} - \tfrac{5}{6} \, \text{m}^2 - \text{m}^3 - \tfrac{1319}{288} \, \text{m}^4 - \tfrac{67}{144} \, \text{m}^5 - \tfrac{2879}{1296} \, \text{m}^6 - \tfrac{1321}{1296} \, \text{m}^7\right] ,$$

as there stated.

We propose now to reduce the preceding formulæ to numerical results. For this purpose we assume

$$n = 17325594''.06085,$$
$$n' = 1295977''.41516,$$

which give

$$\text{m} = \frac{n'}{n - n'} = 0.08084\,89338\,08312,$$
$$\text{m}^2 = 0.00653\,65500\,97941,$$
$$\text{m}^3 = 0.00052\,84731\,06203,$$
$$\text{m}^4 = 0.00004\,27264\,87183,$$
$$\mathfrak{m} = 0.08308\,81293\,65.$$

The numerical value of $\mathfrak{m}$ being substituted in the series, we obtain

$$a_0 = 0.99909\,31419\,62 \left[\frac{\mu}{n^2}\right]^{\frac{1}{3}},$$

$$\begin{aligned}
r \cos v = a_0 [1 &- 0.00718\,00394\,55 \cos 2\tau \\
&+ 0.00000\,60424\,59 \cos 4\tau \\
&+ 0.00000\,00325\,76 \cos 6\tau \\
&+ 0.00000\,00001\,80 \cos 8\tau], \\
r \sin v = a_0 [&\ 0.01021\,14543\,96 \sin 2\tau \\
&+ 0.00000\,57148\,79 \sin 4\tau \\
&+ 0.00000\,00274\,99 \sin 6\tau \\
&+ 0.00000\,00001\,57 \sin 8\tau].
\end{aligned}$$

(To be Continued.)

BIPUNCTUAL COORDINATES.

By F. Franklin, *Fellow of the Johns Hopkins University.*

The expressions for the bilinear coordinates of a point in terms of its trilinear coordinates contain a common denominator which is a linear function of the latter. Whenever, therefore, the trilinear coordinates of a point are such as to make this function equal to zero, its bilinear coordinates are infinite; nor are they infinite under any other supposition: and hence the equation formed by putting this common denominator equal to zero is called the equation of the infinitely distant straight line.

When we examine the corresponding expressions for the bilinear coordinates of a *line*, we do not arrive at any corresponding geometrical idea. These expressions, too, have a common denominator of the first degree; but the equation obtained by putting this denominator equal to zero represents simply the origin of coordinates, a point of no geometrical importance.

I propose to construct a system of coordinates* in which the infinitely distant point shall hold a position similar to that held by the infinitely distant straight line in the bilinear system. In the bilinear system we begin by referring a point to two fixed lines by means of coordinates; we find that the equation of a straight line is of the first degree in the coordinates of its points; and we then define the coordinates of a straight line in such a way that they will be represented by the coefficients of the point-coordinates in its equation when put into a certain form. In the system proposed, a line will be referred to two fixed points by means of coordinates; we shall find that the equation of a point is of the first degree in the coordinates of its lines; and we shall define the coordinates of a point in such a way that they will be represented by the coefficients of the line-coordinates in its equation when put into a certain form, precisely like the corresponding form of the equation of a line in the bilinear system.

*When I had written the greater part of this paper, I accidentally discovered, through one of the notes appended to Salmon's "Conic Sections," that the system of *tangential* coordinates here presented had been used before; where, or to what extent, I do not know.

Plate II.

Fig. 1

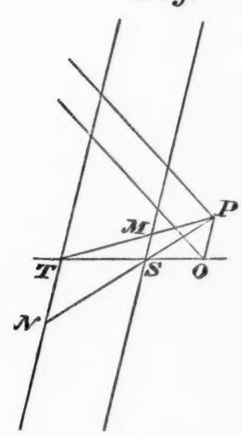


Fig. 2

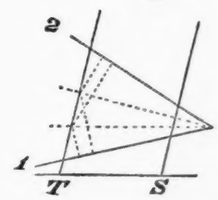


Fig. 3

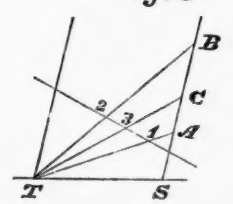


Fig. 4

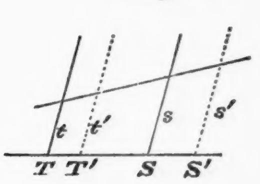


Fig. 5

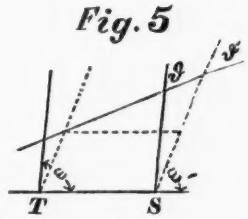


Fig. 6

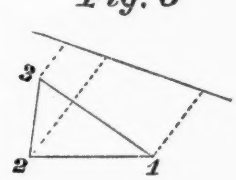


Fig. 7

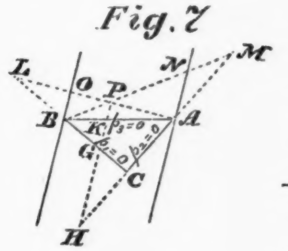


Fig. 8

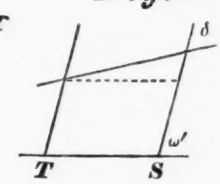


Fig 9

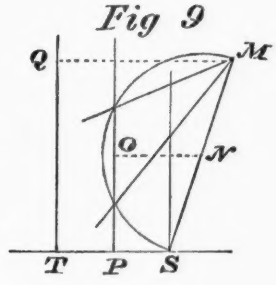
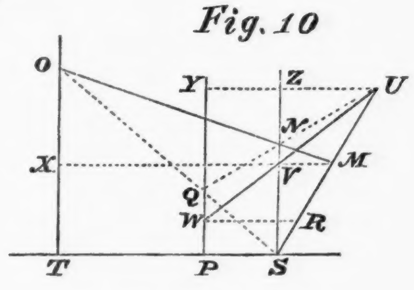


Fig. 10



FRANKLIN, *Bipunctual Coordinates.*

Let the points S, T, (Fig. 1) to which we are to refer all lines, be called the *initials*, and the straight line joining them the *base*.

The coordinates s, t, of a line are its distances from the fixed points S, T, measured in a fixed direction—the same for both initials—which will be called the direction of reference; the lines passing through S and T in the direction of reference may be called the lines of reference.

The equation of any point O on the base is obviously $\frac{s}{t} = \frac{SO}{TO} = c$. Take any point P not on the base. Draw PO parallel to the lines of reference, and let $OP = d$. Then for any line of P, we have $\frac{s-d}{t-d} = c$; this, then, is the equation of P. It is an equation of the first degree; hence the equation of every point is of the first degree.

Conversely, every equation of the first degree is the equation of a point. For the equation above obtained can be written $s = ct + (1-c)\,d$; and any equation of the first degree $As + Bt + C = 0$ can be written $s = -\frac{B}{A}t - \frac{C}{A}$; so that, to obtain the point represented by $As + Bt + C = 0$, we have only to take a point O on the base, such that $\frac{SO}{TO} = -\frac{B}{A}$, and from O to lay off, in the direction of reference, a distance $-\dfrac{\frac{C}{A}}{1+\frac{B}{A}}$ or, $-\dfrac{C}{A+B}$.

There must, of course, be an understanding as to signs. When s or t is measured upward from the initial, it will be regarded as positive; when downward, negative. When the distance SO or TO is measured from the initial in the direction TS, it will be regarded as positive; when in the direction ST, negative. Thus for all points between S and T the ratio $\frac{SO}{TO}$ will be negative; for all points of the base outside of $\overline{ST}$ it will be positive. The above equations of points are in accord with this convention as to signs.

I will venture to make a slight innovation in mathematical language. I have spoken above of the lines of or on a point, meaning the lines passing through a point; for it seems to me that a closer analogy in our language respecting the point and the straight line would tend to facilitate both the comprehension and the remembrance of their geometrical analogies. For this reason, I propose to call the straight line joining two points their *junction;*

for the word *intersection* is used to designate the common point of two straight lines, and we ought to have a corresponding word, *junction*, to designate the common line of two points. And as the word *intersection* is applied to the common points of curves in general, so the word *junction* should be applied to the common lines of curves in general. Thus such expressions as "the tangents drawn from a point to a curve," "the common tangents of two curves," &c., would be replaced by the expressions "the junctions of a point with a curve," "the junctions of two curves," &c.

For the intersection of two lines s_1, t_1, and s_2, t_2, we have the equations $As + Bt + C = 0$, $As_1 + Bt_1 + C = 0$, $As_2 + Bt_2 + C = 0$, whence, eliminating A, B and C, we have for the equation of the point,

$$\begin{vmatrix} s, & t, & 1 \\ s_1, & t_1, & 1 \\ s_2, & t_2, & 1 \end{vmatrix} = 0,$$

If we take as the two lines the junctions of the point with the initials, we have (Fig. 1) $s_1 = SM = a$, $t_1 = 0$; $s_2 = 0$, $t_2 = TN = b$; and the equation of the point becomes

$$\begin{vmatrix} s, & t, & 1 \\ a, & 0, & 1 \\ 0, & b, & 1 \end{vmatrix} = ab - bs - at = 0,$$

or $\frac{s}{a} + \frac{t}{b} - 1 = 0$. If we put $-\frac{1}{a} = p$, $-\frac{1}{b} = q$, this equation becomes $ps + qt + 1 = 0$. Let us call p and q the coordinates of the point P; then we may regard our equation as expressing either the condition that a line s, t, should pass through a fixed point p, q, or the condition that a point p, q, should lie on a fixed line s, t. Thus the form of the combined equation of point and line is precisely the same as in the bilinear system; and we can save time and space by adopting at once such of the formulæ obtained for the bilinear system as depend simply upon the form of this equation. Thus we have:

For any line on the intersection of two lines s_1, t_1; s_2, t_2;

$$s = \frac{s_1 + \lambda s_2}{1 + \lambda}, \; t = \frac{t_1 + \lambda t_2}{1 + \lambda}.$$

Hence $\lambda = \frac{s - s_1}{s_2 - s} = \frac{t - t_1}{t_2 - t}$.

For any point on the junction of two points* p_1, q_1; p_2, q_2;

$$p = \frac{p_1 + \lambda p_2}{1 + \lambda}, \; q = \frac{q_1 + \lambda q_2}{1 + \lambda}.$$

Hence $\lambda = \frac{p - p_1}{p_2 - p} = \frac{q - q_1}{q_2 - q}$.

*It is hardly necessary to mention that the equation of the junction of two points p_1, q_1; p_2, q_2; is

$$\begin{vmatrix} p, & q, & 1 \\ p_1, & q_1, & 1 \\ p_2, & q_2, & 1 \end{vmatrix} = 0.$$

In the case of the line we see at once (Fig. 2) that λ is *proportional* to the distance-ratio of the movable line from the two fixed lines; *i. e.*, the ratio of the perpendiculars dropped upon the two fixed lines from any point of the movable line.

In the case of the point, the meaning of λ is not quite so obvious. Representing by α, β, the distances SA, SB, (Fig. 3) cut off on the S line of reference by the junctions of 1 and 2 respectively with the initial T, and by ρ the corresponding distance SC for any third point 3 of the line 12, we have $p_1 = -\frac{1}{\alpha}$, $p_2 = -\frac{1}{\beta}$, $p = -\frac{1}{\rho}$; so that

$$\lambda = \frac{p - p_1}{p_2 - p} = \frac{\frac{1}{\alpha} - \frac{1}{\rho}}{\frac{1}{\rho} - \frac{1}{\beta}} = \frac{\rho - \alpha}{\beta - \rho} \cdot \frac{\beta}{\alpha} = \frac{\beta}{\alpha} \cdot \frac{AC}{CB}.$$

Now, wherever the point 3 be taken, $\frac{AC}{CB}$ is equal to $\frac{(13)}{(32)}$ multiplied by a constant; so that λ, which is equal to $\frac{AC}{CB}$ multiplied by a constant, is *proportional* to the distance-ratio of the movable point from the two fixed points.

We have thus seen that the parameter λ has the same geometrical significance as the corresponding parameter in the bilinear system; it is needless, therefore, to state the theorems respecting the equations of straight lines passing through a point and of points lying on a straight line, and respecting their anharmonic ratio; the theorems and formulæ for the bipunctual system will be the same as those for the bilinear system, and we can make use of those obtained for the latter whenever we have occasion for them.

I shall consider only two metrical problems relating to the point and straight line; and these because they will be necessary in the transformation of coordinates; and here I shall introduce two words which may be convenient. The distance from a point to a straight line, measured in the direction of reference, will be called the *departure* of the point from the line or of the line from the point; the distance measured on a line parallel to the base will be called the *remove* of the point from the line or of the line from the point. Both these distances will be regarded as positive when the *line is on the positive side of the point.*

1. Required the departure of a line whose coordinates are given from a point whose equation is given.

Let s', t', be the coordinates of the line, and $as + bt + c = 0$ the equation of the point; represent the required departure by δ. The coordinates of the line passing through the given point and parallel to the given line are $s' - \delta$, $t' - \delta$, so that we have

$$a(s' - \delta) + b(t' - \delta) + c = 0; \text{ whence } \delta = \frac{as' + bt' + c}{a + b}$$

2. Required the departure of a point whose coordinates are given from a line whose equation is given.

Let p', q', be the coordinates of the point, and $ap + bq + c = 0$ the equation of the line; then the coordinates of the line are $\frac{a}{c}$ and $\frac{b}{c}$, and the equation of the point is $p's + q't + 1 = 0$: we have, therefore, from the previous case,

$$\delta = \frac{p'\frac{a}{c} + q'\frac{b}{c} + 1}{p' + q'} = \frac{ap' + bq' + c}{c(p' + q')}$$

TRANSFORMATION OF COORDINATES.

I. As long as the direction of reference remains fixed, the only change that can be made in a system of bipunctual coordinates is an alteration in the position of the initials, and any such alteration can be effected by first moving them along the base and then moving them in the direction of reference, by which last process the base itself is moved.

First, let the initials move along the base. Represent by L the distance ST (Fig. 4) between the old initials, by l the distance $S'T'$ between the new initials, by m and n the removes of the new reference-line at S' from S and T respectively, and by m' and n' the corresponding removes of the new reference-line at T'. These quantities are connected by the equations $n - m = n' - m' = L$, $m - m' = n - n' = l$. We have

$$s = s' - \frac{m}{l}(s' - t') = \frac{mt' - m's'}{l}, \quad t = t' - \frac{n'}{l}(s' - t') = \frac{nt' - n's'}{l}.$$

Secondly, let the initials move in the direction of reference. Representing by a and b the departures of the new base from the old initials, we have obviously $s = s' + a$, $t = t' + b$.

II. When the direction of reference is altered, we have, (Fig. 5)

$$\frac{s'}{s} = \frac{t'}{t} = \frac{\sin \vartheta}{\sin \vartheta'} = \frac{\sin(\vartheta' + \omega - \omega')}{\sin \vartheta'} = \cos(\omega - \omega') + \frac{\sin(\omega - \omega')}{\tan \vartheta'}.$$

Now $$\frac{s'-t'}{L}=\frac{\sin(\omega'-\vartheta')}{\sin\vartheta'}=-\cos\omega'+\frac{\sin\omega'}{\tan\vartheta'},$$ whence

$$\frac{1}{\tan\vartheta'}=\frac{s'-t'+L\cos\omega'}{L\sin\omega'},$$ so that we have

$$\frac{s'}{s}=\frac{t'}{t}=\frac{L\left[\sin\omega'\cos(\omega-\omega')+\cos\omega'\sin(\omega-\omega')\right]+(s'-t')\sin(\omega-\omega')}{L\sin\omega'}$$

$$=\frac{L\sin\omega+(s'-t')\sin(\omega-\omega')}{L\sin\omega'},$$

or, $$s=s'\frac{L\sin\omega'}{L\sin\omega+(s'-t')\sin(\omega-\omega')},\quad t=t'\frac{L\sin\omega'}{L\sin\omega+(s'-t')\sin(\omega-\omega')}.$$

If the original system is rectangular and $L=1$, these equations become

$$s=\frac{s'}{\operatorname{cosec}\omega'+(s'-t')\cot\omega'},\qquad t=\frac{t'}{\operatorname{cosec}\omega'+(s'-t')\cot\omega'}.$$

The formulæ for point-coordinates can be immediately obtained from those for line-coordinates by observing the effect of the transformation of the latter upon the equation $ps+qt+1=0$.

TRIPUNCTUAL COORDINATES.

Just as the non-homogeneous equations of the bilinear system are replaced by homogeneous equations when we employ three lines of reference instead of two, the equations of the bipunctual system are replaced by homogeneous equations when we employ three points of reference.

We may define tripunctual coordinates as follows:

The coordinates of a line are three numbers which are to each other as the departures of the line from the three vertices of the triangle of reference, multiplied each by an arbitrary constant.

The coordinates of a point are three numbers which are to each other as the departures of the point from the three sides of the triangle of reference, multiplied each by an arbitrary constant.

Let us designate by s_1, s_2, s_3; p_1, p_2, p_3, the coordinates of line and point, respectively; by m_1, m_2, m_3, the departures of a line from the vertices 1, 2, 3, of the triangle of reference; and by n_1, n_2, n_3, those of a point from the sides 1, 2, 3; then we have

$$\mu s_1=\phi_1 m_1,\quad \mu s_2=\phi_2 m_2,\quad \mu s_3=\phi_3 m_3;\quad \nu p_1=\psi_1 n_1,\quad \nu p_2=\psi_2 n_2,\quad \nu p_3=\psi_3 n_3.$$

These tripunctual coordinates are proportional to trilinear coordinates; so that from any equation in trilinear coordinates we can obtain an equation

in tripunctnal coordinates by replacing $u_1, u_2, u_3, x_1, x_2, x_3$, by $s_1, s_2, s_3, p_1, p_2, p_3$. For, in the first place, representing by M_1, M_2, M_3, the *perpendicular* distances of a straight line from the vertices of the triangle of reference, we have $m_1 : m_2 : m_3 :: M_1 : M_2 : M_3$; and therefore if $\sigma u_i = \lambda_i M_i$ $(i = 1, 2, 3)$, we have only to take $\phi_i = k\lambda_i$ in order to have $s_1 : s_2 : s_3 :: u_1 : u_2 : u_3$. And secondly, representing by N_1, N_2, N_3, the *perpendicular* distances of a point from the sides of the triangle of reference, we have $N_1 = \alpha n_1$, $N_2 = \beta n_2$, $N_3 = \gamma n_3$, where α, β, and γ are constants (viz., the sines of the angles made by the sides of the triangle with a line drawn in the direction of reference); so that, if $\rho x_i = \varkappa_i N_i$, we have $\rho x_1 = \alpha\varkappa_1 n_1$, $\rho x_2 = \beta\varkappa_2 n_2$, $\rho x_3 = \gamma\varkappa_3 n_3$, and we have only to take $\psi_1 = k\alpha\varkappa_1$, $\psi_2 = k\beta\varkappa_2$, $\psi_3 = k\gamma\varkappa_3$, in order to have $p_1 : p_2 : p_3 :: x_1 : x_2 : x_3$.

Let us now obtain the equations connecting tripunctual with bipunctual coordinates.

Let the equations of the vertices of the triangle of reference be

$$\left.\begin{array}{l} a_1 s + b_1 t + c_1 = 0 \\ a_2 s + b_2 t + c_2 = 0 \\ a_3 s + b_3 t + c_3 = 0 \end{array}\right\}$$

where the determinant r, of the coefficients, is not zero.

Let the equations of the sides of the triangle of reference be

$$\left.\begin{array}{l} A_1 p + B_1 q + C_1 = 0 \\ A_2 p + B_2 q + C_2 = 0 \\ A_3 p + B_3 q + C_3 = 0 \end{array}\right\}$$

where the determinant R, of the coefficients, is not zero.

The coefficients $A_1 \ldots . C_3$ are equal (or at least proportional) to the corresponding minors in the determinant of the coefficients $a_1 \ldots . c_3$. We have (page 151)

for any line s, t,

$$m_1 = \frac{a_1 s + b_1 t + c_1}{a_1 + b_1},$$

$$m_2 = \frac{a_2 s + b_2 t + c_2}{a_2 + b_2},$$

$$m_3 = \frac{a_3 s + b_3 t + c_3}{a_3 + b_3}.$$

Taking $\phi_1 = a_1 + b_1$, $\phi_2 = a_2 + b_2$, $\phi_3 = a_3 + b_3$, we have $s_1 : s_2 : s_3 :: a_1 s + b_1 t + c_1 : a_2 s + b_2 t + c_2 : a_3 s + b_3 t + c_3$, or,

$$\mu s_1 = a_1 s + b_1 t + c_1$$
$$\mu s_2 = a_2 s + b_2 t + c_2$$
$$\mu s_3 = a_3 s + b_3 t + c^3$$

and, solving for s and t,

$$s = \frac{A_1 s_1 + A_2 s_2 + A_3 s_3}{C_1 s_1 + C_2 s_2 + C_3 s_3}$$

$$t = \frac{B_1 s_1 + B_2 s_2 + B_3 s_3}{C_1 s_1 + C_2 s_2 + C_3 s_3}$$

for any point p, q,

$$n_1 = \frac{A_1 p + B_1 q + C_1}{C_1 (p + q)},$$

$$n_2 = \frac{A_2 p + B_2 q + C_2}{C_2 (p + q)},$$

$$n_3 = \frac{A_3 p + B_3 q + C_3}{C_3 (p + q)}.$$

Taking $\psi_1 = C_1$, $\psi_2 = C_2$, $\psi_3 = C_3$, we have $p_1 : p_2 : p_3 :: A_1 p + B_1 q + C_1 : A_2 p + B_2 q + C_2 : A_3 p + B_3 q + C_3$, or,

$$\nu p_1 = A_1 p + B_1 q + C_1$$
$$\nu p_2 = A_2 p + B_2 q + C_2$$
$$\nu p_3 = A_3 p + B_3 q + C_3$$

and, solving for p and q,

$$p = \frac{a_1 p_1 + a_2 p_2 + a_3 p_3}{c_1 p_1 + c_2 p_2 + c_3 p_3}$$

$$q = \frac{b_1 p_1 + b_2 p_2 + b_3 p_3}{c_1 p_1 + c_2 p_2 + c_3 p_3}$$

These equations are precisely the same as those connecting Cartesian with trilinear coordinates; we can, therefore, adopt at once the algebraical consequences of the equations. Thus we have $\mu\nu\,(p_1s_1 + p_2s_2 + p_3s_3) = r\,(ps + qt + 1)$, and the combined equation of point and line is $p_1s_1 + p_2s_2 + p_3s_3 = 0$.

The bipunctual system may be regarded as a special case of the tripunctual system. Take, first, the coordinates of a line (Fig. 6); designate by s and t its departures from the vertices 1 and 2, by τ its departure from the vertex 3, and by ρ the departure of 3 from the line 12. We have $\mu s_1 = \phi_1 s$, $\mu s_2 = \phi_2 t$, $\mu s_3 = \phi_3 \tau$. Take $\phi_1 = 1$, $\phi_2 = 1$, $\phi_3 = \frac{1}{\rho}$; then $\mu s_1 = s$, $\mu s_2 = t$, $\mu s_3 = \frac{\tau}{\rho}$. Now let 3 move in the direction of reference to an infinite distance; the limit of the ratio $\frac{\tau}{\rho}$ is unity, and we have $s_1 : s_2 : s_3 :: s : t : 1$.

Secondly, take the coordinates of a point. The tripunctual coordinates of the point P (Fig. 7) are given by $\nu p_1 = \psi_1 n_1$, $\nu p_2 = \psi_2 n_2$, $\nu p_3 = \psi_3 n_3$, where $n_1 = -PG$, $n_2 = -PH$, $n_3 = -PK$. Take $\psi_1 = \frac{1}{AC}$, $\psi_2 = \frac{1}{BC}$, $\psi_3 = -1$, so that $\nu p_1 = \frac{n_1}{AC}$, $\nu p_2 = \frac{n_2}{BC}$, $\nu p_3 = -n_3$. The coordinates of P in the bipunctual system having A and B for its initials (the direction of reference remaining unchanged) are $p = -\frac{1}{AN}$, $q = -\frac{1}{BO}$. We have $PG : OB :: LG : LB$ and $OB : PK :: AB : AK$, whence $PG = PK\frac{AB}{AK}\cdot\frac{LG}{LB}$, $PH : NA :: MH : MA$ and $NA : PK :: AB : BK$, whence $PH = PK\frac{AB}{BK}\cdot\frac{MH}{MA}$; therefore $\nu p_1 = -\frac{PK}{LB}\cdot\frac{AB}{AK}\cdot\frac{LG}{AC}$, $\nu p_2 = -\frac{PK}{MA}\cdot\frac{AB}{BK}\cdot\frac{MH}{BC}$, $\nu p_3 = PK$. Now let C move in the direction of reference to an infinite distance; at the limit, the lines AC, BC, PG, are parallel, LB is OB, MA is NA, and we have $\frac{PK}{LB} = \frac{AK}{AB}$, $\frac{PK}{MA} = \frac{BK}{AB}$, so that $\nu p_1 = -\frac{LG}{AC}$, $\nu p_2 = -\frac{MH}{BC}$, $\nu p_3 = PK$. But at the limit we have also $\frac{LG}{AC} = \frac{BK}{AB} = \frac{PK}{AN}$ and $\frac{MH}{BC} = \frac{AK}{AB} = \frac{PK}{BO}$; so that $\nu p_1 = -\frac{PK}{AN}$, $\nu p_2 = -\frac{PK}{BO}$, $\nu p_3 = PK$; that is, $p_1 : p_2 : p_3 :: p : q : 1$.

If, in the expressions for s and t (p. 154, end), we put the common denominator equal to zero, we obtain the equation of the locus of all lines whose coordinates are infinite; that is, of all infinitely distant lines and all lines drawn in the direction of reference; and since the equation is of the first degree, we must regard this locus as a point: *the infinitely distant point* we may call it for convenience. Regarded analytically, all lines whose coordinates are infinite, pass through the infinitely distant point (*i. e.*, the infinitely distant point proper to the direction of reference); and conversely. Moreover, we must regard *all* infinitely distant points as lying on *one* infinitely distant straight line; for the condition that a point should be infinitely distant is obviously $p = -q$, an equation of the first degree.

We have seen that tripunctual coordinates are perfectly interchangeable with trilinear coordinates; it is needless, therefore, to say anything about transformation from one system of tripunctual coordinates to another. The coefficients of substitution will have the same geometrical meaning as those for trilinear coordinates. The two systems are, in fact, practically identical; whatever can be proved for or with the one can be proved for or with the other; and it is only in their relations to the bipunctual and bilinear systems that the distinction between the tripunctual and the trilinear systems comes into play: tripunctual coordinates standing in the same relation to bipunctual coordinates as trilinear to bilinear.

Defined as anharmonic ratios, the bipunctual coordinates of a line correspond precisely to the bilinear coordinates of a point:

Designate by X the line from which x is measured. The number which represents the x of any point is the anharmonic ratio of four lines passing through the intersection of X with the infinitely distant line; namely, X, the infinitely distant line, the line passing through the point considered, and a fixed line whose position determines the unit of length.

Designate by S the point from which s is measured. The number which represents the s of any line is the anharmonic ratio of four points lying on the junction of S with the infinitely distant point; namely, S, the infinitely distant point, the point lying on the line considered, and a fixed point whose position determines the unit of length.

THE CONIC SECTIONS REFERRED TO BIPUNCTUAL COORDINATES.

Let us see how some of the leading features of curves of the second class, or conics, present themselves when these curves are investigated by means of bipunctual line-coordinates. If we take the infinitely distant point as one vertex of a polar triangle, the opposite side is a diameter of the conic; let us designate this diameter as the base-diameter, and the points lying on it as basics. Any basic being taken as a second vertex of the polar triangle, the third vertex will be another basic harmonically situated with respect to the first basic and the intersections of the diameter with the curve. Two such points will be called conjugate basics; their analogy with conjugate diameters may be set forth thus:

Conjugate basics are two points lying on the base-diameter, so situated that the junction of each with the infinitely distant point is the polar of the other.	Conjugate diameters are two lines passing through the centre, so situated that the intersection of each with the infinitely distant line is the pole of the other.
All conjugate basics are harmonically situated with respect to the basics lying on the junctions of the curve with the infinitely distant point.	All conjugate diameters are harmonically situated with respect to the diameters passing through the intersections of the curve with the infinitely distant line.

Each extremity of the diameter is conjugate to itself; the conjugate of the centre is infinitely distant. The diameter conjugate to the base-diameter is parallel to the lines of reference, and joins the points of contact of the two tangents parallel to the base-diameter.

It will be easy to see what form the equation of the curve assumes when it is referred to conjugate basics. (It is always to be understood, unless otherwise stated, that the direction of reference is that of the conjugate of the base-diameter.) Let the equation of the curve referred to *any* triangle be

$$a_{11}s_1^2 + 2a_{12}s_1s_2 + a_{22}s_2^2 + 2a_{13}s_1s_3 + 2a_{23}s_2s_3 + a_{33}s_3^2 = 0\,.$$

If we put $s_3 = 0$, we shall obtain an equation which is satisfied by the coordinates of the tangents drawn through 3, and also by all lines passing through the intersections of these tangents with the line 12; for the ratio of s_1 to s_2 is constant for all lines passing through a fixed point on the line 12. Now, if

our triangle is a *polar* triangle, these intersections are the points where the line 12 cuts the curve; they are therefore harmonically situated with respect to the points 1 and 2, and the equation which represents them (viz., the equation $a_{11}s_1^2 + 2a_{12}s_1s_2 + a_{22}s_2^2 = 0$, obtained by putting $s_3 = 0$) must be of the form $s_1^2 - \lambda s_2^2 = 0$; that is, the term containing s_1s_2 can not appear. It is evident, in the same way, that the terms containing s_1s_3 and s_2s_3 can not appear; so that the equation of the curve referred to any polar triangle is of the form $a_1s_1^2 + a_2s_2^2 + a_3s_3^2 = 0$. If, then, in this equation, we replace s_1 by s, s_2 by t, s_3 by 1, we shall have the form of the equation of the curve referred to conjugate basics which may therefore be written $\frac{s^2}{c^2} + \frac{t^2}{d^2} = 1$.* Either c^2 or d^2 may be negative, *i. e.*, either c or d imaginary; they cannot both be negative unless the curve itself is imaginary.

We see that for any value of s there are two equal and opposite values of t; *i. e.*, the two tangents drawn from any point of either line of reference cut the other in two points equidistant from the base. This geometrical property follows immediately from the definition of conjugate basics; and we could have inferred the form of the equation from this property, instead of the converse.

Before going any further, let us see how this simple form of the equation is derived from the most general bipunctual equation of the curve $a_{11}s^2 + 2a_{12}st + a_{22}t^2 + 2a_{13}s + 2a_{23}t + a_{33} = 0$. As tripunctual coordinates are interchangeable with trilinear coordinates, and are replaced by bipunctual coordinates in the same manner as trilinear coordinates are replaced by bilinear coordinates, we can here use at once certain algebraic results obtained for the latter (see Clebsch, p. 82, *et seqq.*) The coordinates of the base-diameter are

$$\sigma = \frac{A_{13}}{A_{33}} = \frac{a_{21}a_{32} - a_{22}a_{31}}{a_{11}a_{22} - a_{12}^2}, \qquad \tau = \frac{A_{23}}{A_{33}} = \frac{a_{31}a_{12} - a_{11}a_{32}}{a_{11}a_{22} - a_{12}^2};$$

and I shall suppose, for the present, that A_{33} is not zero. To transfer the initials to the base-diameter (the initials moving in the direction of reference) we put $s = s' + \sigma$, $t = t' + \tau$, and our equation becomes $a_{11}s'^2 + a_{22}t'^2 + 2a_{12}s't' + \frac{A}{A_{33}} = 0$. (It is supposed *throughout* that A, the determinant of $a_{11} \dots . a_{33}$ is not zero.)

* In the above reasoning, as in general where the case has admitted it, I have followed the method used in Clebsch's "*Vorlesungen über Geometrie.*"

If we represent by m and n the removes of a straight line from the initials S and T, and by ϑ the ratio of the sine of the angle the line makes with the base to the sine of the angle it makes with the lines of reference, we have $s = -m\vartheta$, $t = -n\vartheta$; also $n - m = L$, where L is the distance between the initials. When the line considered is parallel to the base, $\vartheta = 0$; when it is parallel to the lines of reference, $\vartheta = \infty$.

The condition that a line s, t, should pass through the pole of the line s', t', with reference to the curve $a_{11}s^2 + a_{22}t^2 + 2a_{12}st + \frac{A}{A_{33}} = 0$, is $a_{11}ss' + a_{22}tt' + a_{12}(st' + s't) + \frac{A}{A_{33}} = 0$; when the two lines are parallel to the lines of reference, this equation becomes (replacing s by $m\vartheta$, t by $n\vartheta$, s' by $m'\vartheta$, and t' by $n'\vartheta$, and dividing by ϑ, and then making ϑ infinite)

$$a_{11}mm' + a_{22}nn' + a_{12}(mn' + m'n) = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or, substituting $L + m$ for n, $L + m'$ for n',

$$(a_{11} + 2a_{12} + a_{22})\, mm' + (a_{12} + a_{22})(m + m')\, L + a_{22}L^2 = 0\,.$$

If we transfer the initials to points whose distances from the former initials are m, n, and m', n', respectively, we shall find that (1) is the necessary and sufficient condition of the disappearance of the term containing st. To effect the transformation, we replace s by $\frac{mt - m's}{l}$ and t by $\frac{nt - n's}{l}$ (where $l = m - m' = n - n'$ is the distance between the new initials); and the equation becomes $(a_{11}m'^2 + 2a_{12}m'n' + a_{22}n'^2)\, s^2 + (a_{11}m^2 + 2a_{12}mn + a_{22}n^2)\, t^2 - 2\left[a_{11}mm' + a_{22}nn' + a_{12}(mn' + m'n)\right] st + \frac{A}{A_{33}}\, l^2 = 0$. (1) expresses the necessary and sufficient condition of the disappearance of the term containing st; when that condition is fulfilled, our equation becomes $(a_{11}m'^2 + 2a_{12}m'n' + a_{22}n'^2)\, s^2 + (a_{11}m^2 + 2a_{12}mn + a_{22}n^2)\, t^2 + \frac{A}{A_{33}}\, l^2 = 0$, or $\frac{s^2}{c^2} + \frac{t^2}{d^2} = l^2$, where

$$c^2 = -\frac{A}{A_{33}} \cdot \frac{1}{a_{11}m'^2 + 2a_{12}m'n' + a_{22}n'^2}, \text{ and } d^2 = -\frac{A}{A_{33}} \cdot \frac{1}{a_{11}m^2 + 2a_{12}mn + a_{22}n^2}.$$

In the equation $\frac{s^2}{c^2} + \frac{t^2}{d^2} = l^2$, c^2 and d^2 are real quantities, but not necessarily both positive. For $s = t$, we have $\frac{s^2}{c^2} + \frac{s^2}{d^2} = l^2$, whence $s = \pm\, l\sqrt{\frac{c^2d^2}{c^2 + d^2}}$;

hence the length of the diameter conjugate to the base diameter is $2l\sqrt{\dfrac{c^2d^2}{c^2+d^2}}$. Putting $-\mu\vartheta$ for s and $-\nu\vartheta$ for t in the equation of the curve, it becomes $\dfrac{\mu^2\vartheta^2}{c^2}+\dfrac{\nu^2\vartheta^2}{d^2}=l^2$; when $\vartheta=\infty$, we get $\dfrac{\mu^2}{c^2}+\dfrac{\nu^2}{d^2}=0$; or, putting for ν its value $l+\mu$, $\dfrac{\mu^2}{c^2}+\dfrac{(l+\mu)^2}{d^2}=0$, whence $\mu=l\dfrac{-c^2\pm\sqrt{-c^2d^2}}{c^2+d^2}$. The difference between the two values of μ is the length of the base-diameter, which is therefore $\pm 2l\dfrac{\sqrt{-c^2d^2}}{c^2+d^2}$. Representing by $2a$, $2b$, the lengths of the base diameter and its conjugate, respectively, we have, then, $a^2=-l^2\dfrac{c^2d^2}{(c^2+d^2)^2}$, $b^2=l^2\dfrac{c^2d^2}{c^2+d^2}$; whence $c^2+d^2=-\dfrac{b^2}{a^2}$, $c^2d^2=-\dfrac{b^4}{a^2l^2}$, so that c^2 and d^2 are the roots of the equation $x^2+\dfrac{b^2}{a^2}x-\dfrac{b^4}{a^2l^2}=0$, and their values are $-\dfrac{b^2}{2a^2l}(l\pm\sqrt{l^2+4a^2})$.

When $s=0$, $t=\pm ld$; when $t=0$, $s=\pm lc$; that is, the departure from S of a tangent passing through T is lc, and the departure from T of a tangent passing through S is ld; these distances may be called the initial departures of the curve. We have $l^2c^2+l^2d^2=-\dfrac{b^2}{a^2}l^2$, $l^4c^2d^2=-\dfrac{b^4}{a^2}l^2$. Now a and b, being the lengths of the base-diameter and its conjugate, are unaltered when the initials are moved on the base; hence, whenever the initials are conjugate basics on a given diameter, the sum of the squares of the initial departures is proportional to the square of the distance between the initials, and the product of the initial departures is proportional to the distance between the initials. The initial departures cannot both be real unless the initials are on a diameter which cuts the curve in imaginary points and whose conjugate cuts the curve in real points; for it is only when b^2 is positive and a^2 negative that the expressions for c^2 and d^2, namely $-\dfrac{b^2}{2a^2l}(l\pm\sqrt{l^2+4a^2})$, are both positive.

Let us see how our curves can be classified. We have

1) When $c^2+d^2=0$, a and b are both infinite.
2) When c^2+d^2 is negative, a and b are real and finite.
3) When c^2+d^2 is positive, a is imaginary and b real if c^2 and d^2 are both positive; a real and b imaginary if c^2 and d^2 have opposite signs.

(When $c^2 + d^2 = -1$, a and b are equal in absolute value and both real; when $c^2 + d^2 = +1$, a^2 and b^2 are equal in absolute value, but either a or b is imaginary.)

It would seem, then, that we can divide our curves into three classes:

1) Those in which the base-diameter and its conjugate are both infinite in length. $(c^2 + d^2 = 0)$.

2) Those in which they are both terminated in real points $(c^2 + d^2 < 0)$.

3) Those in which one of the two diameters is terminated in real points and the other in imaginary points $(c^2 + d^2 > 0)$.

But we must see whether these characteristics are preserved when the direction of reference is changed; or in other words, whether, if one pair of conjugate diameters belongs to a certain one of these categories, *every* pair of conjugate diameters of the same curve will belong to the same category. Returning to the expressions which we replaced by c^2 and d^2 (page 159), we have

$c^2 + d^2 = -\frac{A}{A_{33}}\left(\frac{1}{a_{11}m^2 + 2a_{12}mn + a_{22}n^2} + \frac{1}{a_{11}m'^2 + 2a_{12}m'n' + a_{22}n'^2}\right)$. The quantity in the parenthesis is equal to $\frac{a_{11}m^2 + 2a_{12}mn + a_{22}n^2 + a_{11}m'^2 + 2a_{12}m'n' + a_{22}n'^2}{(a_{11}m^2 + 2a_{12}mn + a_{22}n^2)(a_{11}m'^2 + 2a_{12}m'n' + a_{22}n'^2)}$.

If we subtract from the denominator of this fraction the square of $a_{11}mm' + a_{22}nn' + a_{12}(mn' + m'n)$, which is equal to zero, and subtract from the numerator $2[a_{11}mm' + a_{22}nn' + a_{12}(mn' + m'n)]$, the fraction reduces to

$$\frac{a_{11}(m - m')^2 + 2a_{12}(m - m')(n - n') + a_{22}(n - n')^2}{(a_{11}a_{22} - a_{12}^2)(mn' - m'n)^2},$$

or, since $m - m' = n - n' = l$ and $n - m = n' - m' = L$, to $\frac{a_{11} + 2a_{12} + a_{22}}{A_{33}L^2}$; so that we have

$$c^2 + d^2 = -\frac{A}{A_{33}^2 L^2}(a_{11} + 2a_{12} + a_{22}).$$

It will be convenient to postpone the investigation of the change which this expression undergoes when the direction of reference is changed; but I must anticipate the results of that investigation so far as to say that A becomes $\alpha^2 A$, where α is a real trigonometrical function which cannot vanish; and that $a_{11} + 2a_{12} + a_{22}$ remains entirely unchanged. So we see—since the denominator is essentially positive—that the sign of the above expression is unal-

tered by a change in the direction of reference; and that if it is zero for any direction of reference, it is zero for all. We have, then, the general classification:

1) $a_{11} + 2a_{12} + a_{22}$ is zero; all diameters have infinite length (parabola).

2) $a_{11} + 2a_{12} + a_{22}$ has the same sign as A; all diameters have finite length (ellipse).

3) $a_{11} + 2a_{12} + a_{22}$ has the sign contrary to that of A; of each pair of conjugate diameters, one has real length and one imaginary length (hyperbola).

We have found $c^2 + d^2 = -\dfrac{A}{A_{33}^2 L^2}(a_{11} + 2a_{12} + a_{22})$, and we have also $c^2d^2 = \dfrac{A^2}{A_{33}^2} \cdot \dfrac{1}{(a_{11}m^2 + 2a_{12}mn + a_{22}n^2)(a_{11}m'^2 + 2a_{12}m'n' + a_{22}n'^2)} = \dfrac{A^2}{A_{33}^3 L^2 l^2}$. For any given direction of reference, all the quantities involved in these values, except l, are constant; so that we see again that $(c^2 + d^2)\, l^2$ is proportional to l^2 and that cdl^2 is proportional to l.

Let us investigate the condition of the equality of conjugate diameters. That condition is expressed (see p. 161, l. 1) by the equation $-\dfrac{A}{A_{33}^2 L^2}(a_{11} + 2a_{12} + a_{22}) = -1$. If we represent by ϕ the angle made by the *original* base with the lines of reference, and call the distance between the original initials unity, we have, since $\dfrac{A_{13}}{A_{33}}$ and $\dfrac{A_{23}}{A_{33}}$ are the coordinates of the base-diameter, $L^2 = 1 + \left(\dfrac{A_{13} - A_{23}}{A_{33}}\right)^2 + 2\,\dfrac{A_{13} - A_{23}}{A_{33}} \cos\phi$, and the equation of condition becomes $\dfrac{A\,(a_{11} + 2a_{12} + a_{22})}{A_{33}^2 + (A_{13} - A_{23})^2 + 2A_{33}(A_{13} - A_{23})\cos\phi} = 1$.— Let the equation of the curve referred to a rectangular system be

$$a_{11}s^2 + 2a_{12}st + a_{22}t^2 + 2a_{13}s + 2a_{23}t + a_{33} = 0.$$

Let the direction of reference be altered so that the lines of reference shall make an angle ω' with the base; denote the coefficients of the new equation by a'_{11}, &c., their determinant by A', and its minors by A'_{11}, &c. The transformation is effected by means of the formulæ (p. 153) $s = \dfrac{s'}{\alpha + \beta\,(s' - t')}$,

$t = \dfrac{t'}{\alpha + \beta(s' - t')}$; where $\alpha = \dfrac{1}{\sin\omega'}$ and $\beta = \dfrac{\cos\omega'}{\sin\omega'}$, so that $\cos\omega' = \dfrac{\beta}{\alpha}$ and $\alpha^2 = 1 + \beta^2$. We have

$$a'_{11} = a_{11} + 2a_{13}\beta + a_{33}\beta^2, \quad a'_{22} = a_{22} - 2a_{23}\beta + a_{33}\beta^2, \quad a'_{33} = a_{33}\alpha^2,$$
$$a'_{12} = a_{12} - (a_{13} - a_{23})\beta - a_{33}\beta^2, \quad a'_{13} = a_{13}\alpha + a_{33}\alpha\beta, \quad a'_{23} = a_{23}\alpha - a_{33}\alpha\beta;$$

whence we find

$$A' = \begin{vmatrix} a_{11} + 2a_{13}\beta + a_{33}\beta^2, & a_{12} - (a_{13} - a_{23})\beta - a_{33}\beta^2, & a_{13}\alpha + a_{33}\alpha\beta \\ a_{12} - (a_{13} - a_{23})\beta - a_{33}\beta^2, & a_{22} - 2a_{23}\beta + a_{33}\beta^2, & a_{23}\alpha - a_{33}\alpha\beta \\ a_{13}\alpha + 0 + a_{33}\alpha\beta, & a_{23}\alpha + 0 - a_{33}\alpha\beta, & 0 + a_{33}\alpha^2 \end{vmatrix} = \alpha^2 A,$$

$$A'_{33} = A_{33} - 2\beta(A_{13} - A_{23}) + \beta^2(A_{11} + A_{22} - 2A_{12}) = A_{33} - 2\beta G + \beta^2 H,$$
$$A'_{13} - A'_{23} = -\alpha\beta(A_{11} + A_{22} - 2A_{12}) + \alpha(A_{13} - A_{23}) = \alpha G - \alpha\beta H,$$

(where, for brevity, we put $A_{13} - A_{23} = G$, $A_{11} + A_{22} - 2A_{12} = H$) and $a'_{11} + 2a'_{12} + a'_{22} = a_{11} + 2a_{12} + a_{22}$. If, now, in the equation

$$\frac{A'(a'_{11} + 2a'_{12} + a'_{22})}{A'^2_{33} + (A'_{13} - A'_{23}) + 2A'_{33}(A'_{13} - A'_{23})\cos\phi} = 1,$$

we substitute the values above obtained, and substitute $\dfrac{\beta}{\alpha}$ ($= \cos\omega'$) for $\cos\phi$, we obtain, after reduction,

$$A^2_{33} + G^2 - A(a_{11} + 2a_{12} + a_{22}) - 2G(A_{33} + H)\beta$$
$$+ [G^2 + H^2 - A(a_{11} + 2a_{12} + a_{22})]\beta^2 = 0.$$

From this equation we can, in general, obtain two values of β, which give us the directions of the two equal conjugate diameters.

In order that *every* diameter should be equal to its conjugate, the above equation must be satisfied for all values of β, and we must have

$$A^2_{33} + G^2 - A(a_{11} + 2a_{12} + a_{22}) = 0, \quad G(A_{33} + H) = 0,$$
$$\text{and } G^2 + H^2 - A(a_{11} + 2a_{12} + a_{22}) = 0.$$

Substituting for G its value, and remembering that $Aa_{11} = A_{22}A_{33} - A^2_{23}$, &c., the first equation becomes $2(A_{13} - A_{23})^2 - A_{33}(A_{11} - 2A_{12} + A_{22} - A_{33}) = 0$, or $2G^2 - A_{33}(H - A_{33}) = 0$, and the third becomes likewise $2G^2 - A_{33}H + H^2 = 0$, or $2G^2 + H(H - A_{33}) = 0$. We have, then,

$$(1) \ldots .\ 2G^2 - A_{33}(H - A_{33}) = 0; \qquad (2) \ldots .\ G(H + A_{33}) = 0;$$
$$(3) \ldots .\ 2G^2 + H(H - A_{33}) = 0.$$

Equation (2) shows that we must have either $G = 0$ or $H + A_{33} = 0$. In the first case, we must also have $H - A_{33} = 0$ in order that equations (1) and

(3) should be satisfied; in the second case, equations (1) and (3) become $G^2 + A_{33}^2 = 0$, which cannot be satisfied if the coefficients of the given equation are real, as we suppose them to be throughout. The necessary and sufficient condition, therefore, that every diameter should be equal to its conjugate is, in any rectangular system of coordinates, (the distance between the initials being taking as unity)

$$G = 0 \text{ and } H - A_{33} = 0; \text{ that is, } A_{13} = A_{23} \text{ and } A_{11} - 2A_{12} + A_{22} = A_{33}.$$

The square of half the diameter making the angle $\cot^{-1}\beta$ with the original base is (pp. 160, 162)

$$l^2\frac{c^2d^2}{c^2 + d^2} = -\frac{A'}{A'_{33}(a'_{11} + 2a'_{12} + a'_{22})} = -\frac{a^2 A}{(A_{33} - 2\beta G + \beta^2 H)(a_{11} + 2a_{12} + a_{22})};$$

when $G = 0$ and $H = A_{33}$, this becomes

$$-\frac{a^2 A}{(1 + \beta^2) A_{33}(a_{11} + 2a_{12} + a_{22})} = -\frac{A}{A_{33}(a_{11} + 2a_{12} + a_{22})},$$

an expression independent of β. So we see that when every diameter is equal to its conjugate, all diameters are equal; *i. e.*, the curve is a circle. The condition $A_{13} = A_{23}$ shows that the base-diameter is parallel to the original base; we have seen that this condition holds in the case of the circle for *any* rectangular system: therefore, in the circle, every diameter is perpendicular to its conjugate.

The condition $c^2 + d^2 = +1$ (see p. 161, l. 2) would become, in the same way,

$$-(1+\beta^2) A (a_{11} + 2a_{12} + a_{22}) = A_{33}^2 + G^2 - 2G(A_{33} + H)\beta + (G^2 + H^2)\beta^2,$$

or

$$A_{33}^2 + G^2 + A(a_{11} + 2a_{12} + a_{22}) - 2G(A_{33} + H)\beta + [G^2 + H^2 + A(a_{11} + 2a_{12} + a_{22})]\beta^2 = 0;$$

and we should find, in like manner, that if this equation is satisfied for all values of β we have $A_{33}(H + A_{33}) = 0$, $G(H + A_{33}) = 0$, and $H(H + A_{33}) = 0$, so that the only condition here is $H + A_{33} = 0$.

It may be worth while to observe that the results obtained for A', A'_{33}, $A'_{13} - A'_{23}$, and $a'_{11} + 2a'_{12} + a'_{22}$, are entirely independent of the fact that the original system was rectangular; they apply to all cases: and it is only in the replacing of a^2 by $1 + \beta^2$ and of $\cos\phi$ by $\frac{\beta}{a}$, that the rectangularity of the original system comes into play, and simplifies the results.

The angle between the base-diameter and its conjugate (Fig. 8) is given by

$$\frac{\sin\delta}{\sin(\omega'-\delta)}=\frac{1}{\frac{A'_{13}}{A'_{33}}-\frac{A'_{23}}{A'_{33}}}=\frac{A'_{33}}{A'_{13}-A'_{23}};\ \text{whence}\ \tan\delta=\frac{A'_{33}\sin\omega'}{A'_{13}-A'_{23}+A'_{33}\cos\omega'}$$

$$=\frac{A'_{33}}{(A'_{13}-A'_{23})\,\alpha+A'_{33}\beta}=\frac{A_{33}-2\beta G+\beta^2H}{\alpha^2G-\alpha^2\beta H+\beta A_{33}-2\beta^2G+\beta^3H}$$

$$=\frac{A_{33}-2\beta G+\beta^2H}{(1-\beta^2)\,G-\beta H+\beta A_{33}};$$ so that we have the equation

$$\beta^2(H+G\tan\delta)+\beta\,[(H-A_{33})\tan\delta-2G]-G\tan\delta+A_{33}=0.$$

The condition for the reality of the values of β is

$$4\,(A_{33}-G\tan\delta)(H+G\tan\delta)\ \overline{\overline{<}}\ [(H-A_{33})\tan\delta-2G]^2,$$

which reduces to

$$\tan^2\delta\ \overline{\overline{>}}\ 4\,\frac{A_{33}H-G^2}{(H-A_{33})^2+4G}\,.$$

This condition is satisfied for all values of δ if $A_{33}H-G^2$ is not positive, *i. e.* if $A_{33}\,(A_{11}-2A_{12}+A_{22})-(A_{13}-A_{23})^2\ \overline{\overline{<}}\ 0$, and this can be put into the form $A_{11}A_{33}-A_{13}^2+A_{22}A_{33}-A_{23}^2+2\,(A_{13}A_{23}-A_{12}A_{33})\ \overline{\overline{<}}\ 0$, or $A\,(a_{11}+2a_{12}+a_{22})\ \overline{\overline{<}}\ 0$. Thus we see that in the parabola and hyperbola the conjugate diameters may make any angle with each other, but that in the ellipse there is a minimum angle given by the equation $\tan^2\delta=4\,\frac{A_{33}H-G^2}{(H-A_{33})^2+4G^2}$. In the case of the circle, $H-A_{33}=0$ and $G=0$, and this equation becomes $\tan^2\delta=\infty$, whence we see again that in the circle every diameter is perpendicular to its conjugate.

Solving the quadratic in β found above, we obtain

$$\beta=\frac{-(H-A_{33})\tan\delta+2G\pm\sqrt{[(H-A_{33})^2+4G^2]\tan^2\delta-4\,(A_{33}H-G^2)}}{2(H+G\tan\delta)}.$$

Since $A_{33}H-G^2\equiv A\,(a_{11}+2a_{12}+a_{22})$, we have, in the case of the parabola, $A_{33}H-G^2=0$, and the values of β will be found to reduce to

$$\beta=\frac{A_{33}-G\tan\delta}{G+A_{33}\tan\delta}\ \text{and}\ \beta=\frac{A_{33}}{G}.$$

The second of these values is independent of δ; therefore in the parabola there is a fixed direction in which one diameter in every pair runs. In order that this direction should coincide with the direction of reference we must have

$\beta = 0$, whence $A_{33} = 0$. Now, when $A_{33} = 0$, the coordinates of the base-diameter are infinite; and since we have seen that the conjugate diameters may make any angle with each other, this cannot be interpreted to mean that the base-diameter is parallel to the lines of reference, *i. e.* to its conjugate: it must, then, be infinitely distant, and we have therefore found that in the parabola all diameters that are not infinitely distant are parallel to each other.

Let the equation of a conic referred to a certain pair of conjugate basics be $\frac{s^2}{C^2} + \frac{t^2}{D^2} = L^2$, and its equation, referred to any other pair of conjugate basics on the same diameter, $\frac{s^2}{c^2} + \frac{t^2}{d^2} = l^2$. Putting $C^2 + D^2 = g$, we know that, wherever the new basics be taken, we have $c^2 + d^2 = g$ and $l^2c^2d^2 = L^2C^2D^2$; and if we assign any particular value to l, c, or d, we can obtain the values of the other two of these quantities from these equations. If we put $d^2 = -1$, so that the equation of the curve becomes $s^2 = c^2(t^2 + l^2)$, we have $c^2 = g + 1$, $l^2 = -\frac{C^2D^2}{g+1}L^2$. The condition that the basics for which the equation of the curve assumes this form should be real points is, that this value of l^2 should be positive; and it is easy to see when this condition is fulfilled:

In the parabola, C^2D^2 is negative and $g = 0$; hence l^2 is positive.

In the ellipse, C^2D^2 is negative and g is negative; so that l^2 is positive when g is numerically less than 1, and negative when g is numerically greater than 1; that is (p. 160), positive on a diameter that is greater than its conjugate and negative on one that is smaller than its conjugate.

In the hyperbola g is always positive; C^2D^2 is negative when the diameter cuts the curve, and positive when it does not; in the former case l^2 is positive, in the latter, negative.

Of course, to every numerical value of l there correspond two pairs of conjugate basics, symmetrically situated with respect to the centre; in the parabola, one of these pairs is infinitely distant.

It appears from the discussion of the angles made by diameters with their conjugates, that every conic has one pair of rectangular conjugate diameters, or axes; and we now see that on *one* of the axes there can be found, on each side of the centre, a pair of conjugate basics for which the equation of the curve assumes the form $s^2 = e^2(t^2 + l^2)$; that is, a pair of

conjugate basics, S and T, such that *the ratio of the distances from S to the intersections of any tangent with the S and T lines is constant.* It is plain that e^2 is always positive, and that in the parabola $e^2 = 1$, in the ellipse $e^2 < 1$, and in the hyperbola $e^2 > 1$.

Let us see how, by means of this property, we can construct the tangents to a conic from a given point.

For the parabola, the construction is very simple. Let S and T (Fig. 9) be the basics in question, and M the given point. The line drawn from S to the point midway between the intersections of the tangent with the S and T lines is perpendicular to the tangent (since S is equidistant from those intersections); that point is consequently on the circle described upon SM as a diameter. But it is also on the line drawn parallel to the S and T lines midway between them; it is therefore at the intersection of this line with the above-mentioned circle. Hence, to construct the tangents from M, we draw OP midway between the S and T lines; and from N, the middle point of MS, we describe a circle with NS as radius; the lines joining M with the intersections of this circle with OP are the required tangents. If $NO = NS$, the tangents coincide and M is a point of the parabola; and conversely, if M is a point of the parabola, the tangents coincide and $NO = NS$. But $\frac{NO}{NS} = \frac{MQ}{MS}$; therefore, the distance of any point of the parabola from S is equal to its distance from the T line. When MQ is greater than MS, the two tangents are imaginary; when MQ is less than MS, the two tangents are real and distinct; when M is on the T line, the two tangents are perpendicular to each other. We may call S the focus and the T line the directrix of the parabola.

Let us proceed to the ellipse and hyperbola. We have, for any tangent MO (Fig. 10), $\frac{SN}{SO} = e$. Denote the distance ST by Ef, where $E = \frac{1}{e}$, take $SP = f$, and draw PQ parallel to the lines of reference; it intersects SO at a point Q. It is obvious that $SQ = SN$, and QN is a tangent to the parabola whose focus is S and directrix PQ. If, therefore, we can determine the point U where QN intersects SM, we can construct the tangents from M by constructing the tangents from U to the parabola (SP) and joining M with the intersections of those tangents with the S line. Denote the distance SM by Ez and take $SR = z$. RQ is parallel to MO; hence, if we draw *any* line MV, meeting the S line at V, and a parallel line RW meeting PQ at W, U is the

intersection of VW with SM. We have, then, the following construction for the tangents to the ellipse (or hyperbola) from the point M, the curve being given by S, T, and P. Through M draw any line meeting the S line at V; on SM take a point R such that $\frac{SR}{SM} = \frac{SP}{ST}$, and through R draw a line parallel to MV, meeting the P line at W; join VW, and from U, the intersection of VW with SM, construct the tangents to the parabola (SP); the lines joining M with the points where these tangents cut the S line are the required tangents.

Let us take MV and RW *parallel to* ST, and produce MV to meet the T line in X; also draw UY parallel to ST and meeting the P line in Y. It can be shown* that $\frac{US}{UY} = E\frac{MS}{MX}$; whence it follows at once (from what we have found for the parabola) that the distance of any point on the ellipse (or hyperbola) from the focus is equal to e times its distance from the directrix—S being again designated the focus and the T line the directrix; also that no tangent can be drawn when the ratio of these distances is less than e, and two distinct tangents when it is greater than e.

It is plain that in the ellipse the foci lie between the extremities of the axis and the directrices outside of them, and that the reverse is the case with the hyperbola; it follows immediately from the above, therefore, that in the ellipse the sum of the distances of any point from the two foci is constant, and in the hyperbola the difference of the distances of any point from the two foci is constant.

The following are obvious consequences of the property by which the focus and directrix were brought to our notice: in the parabola, any tangent makes, at its intersection with the directrix, equal angles with the directrix and the line joining that intersection with the focus; in the ellipse and hyperbola, any tangent makes, at its intersections with the two directrices, equal angles with the lines joining those intersections with the corresponding foci.

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* Denote MX by Ey; then $RW = y$. We have

$$RU = RM\frac{RW}{RW - MV} = (E-1)\,z\,\frac{y}{Ef - (E-1)\,y}, \qquad SU = SR + RU = z + (E-1)\,z\,\frac{y}{Ef - (E-1)\,y}$$
$$= \frac{Efz}{Ef - (E-1)\,y}; \qquad UZ = SU\frac{MV}{SM} = \frac{Efz}{Ef - (E-1)\,y}\cdot\frac{E\,(y-f)}{Ez} = \frac{Ef\,(y-f)}{Ef - (E-1)\,y}, \qquad UY = UZ + f$$
$$= \frac{fy}{Ef - (E-1)\,y}; \text{ therefore } \frac{SU}{UY} = \frac{Ez}{y} = E\frac{Ez}{Ey} = E\frac{SM}{MX}.$$

Let $\frac{s^2}{C^2} + \frac{t^2}{D^2} = L^2$ be the equation of a conic referred to any pair of conjugate basics on an axis; is there a point on the axis the ratio of whose distances from the intersections of any tangent with the S and T lines is constant? Writing the equation in the form $s^2 = -\frac{C^2}{D^2}t^2 + C^2L^2$, we see that if such a point exists, then, denoting its distance from S by αL (whence its distance from T is $(1+\alpha)\,L$), the above equation must be identical with $s^2 + \alpha^2L^2 = -\frac{C^2}{D^2}[t^2 + (1+\alpha)^2\,L^2]$; so that we have, for determining α, $\frac{C^2}{D^2}(1+\alpha)^2 + \alpha^2 = -C^2$, whence

$$\alpha = \frac{-C^2 \pm \sqrt{-C^2D^2\,(C^2+D^2+1)}}{C^2+D^2} = \frac{-C^2 \pm \sqrt{-C^2D^2\,(g+1)}}{g}.$$

It is plain that the condition of the reality of these values is the same as that of the existence of real foci (see p. 166, l. 14); so that points possessing the property in question can be found only on the axis on which the foci are situated. The distance from S of the point midway between the two points corresponding to the two values of α is $\frac{\alpha_1+\alpha_2}{2}L = -\frac{C^2}{g}L$, which, it will be readily seen from p. 160, l. 4, is the distance of the centre; so that the two points are equidistant from the centre. The distance between the points is $(\alpha_1 - \alpha_2)\,L = \frac{2\sqrt{-C^2D^2L^2\,(g+1)}}{g} = \frac{2\sqrt{e^2l^2\,(g+1)}}{g} = \frac{2e^2l}{g}$ (since $g+1=e^2$); and this is obviously equal, in absolute value, to twice the distance from the focus to the centre; that is, equal to the distance between the foci. Hence, the points sought are the foci themselves, and we have the following theorem (in which the term "conjugate ordinates" is to be understood as meaning ordinates to the transverse axis passing through conjugate basics on that axis):

If M and M' be the points in which *any* tangent to a conic intersects a *fixed* pair of conjugate ordinates, and F a focus of the conic, the ratio $\frac{FM}{FM'}$ is constant, and the same for both foci.*

* This theorem admits of several easy geometrical demonstrations.

It may be remarked that this ratio, $\frac{C^2}{D^2}$, is unity for all pairs of conjugate ordinates in the parabola; in the ellipse and hyperbola it varies for different pairs, from 0 to 1 or from 1 to ∞.

The following are one or two obvious consequences of this proposition:

1. Since the ordinate conjugate to the minor axis is a straight line perpendicular to the transverse axis and infinitely distant, and since the distance from the focus to the intersection of any tangent with this infinitely distant ordinate is proportional to the secant of the angle made by the tangent with the transverse axis; it follows from the theorem just established that the distance from the focus to the intersection of any tangent with the minor axis is proportional to the secant of the angle made by the tangent with the transverse axis. In the case of the ellipse, this distance is equal to half the transverse axis when the angle is 0; so that the distance from the focus to the intersection of any tangent with the minor axis is equal to the distance intercepted by the axes on a line drawn through an extremity of the transverse axis, parallel to the tangent. It can easily be seen that this result is true of the hyperbola also, within the limits to which its real tangents are confined.

2. Let M be the point of contact of a tangent, and M' the point at which it intersects the transverse axis. We have, by the above theorem, $\frac{FM}{FM'} = \frac{F'M}{F'M'}$, whence we see that the angle $F'MM'$ is the supplement of FMM'; that is, the tangent makes equal angles with the lines joining its point of contact with the foci.

Finally, let us ascertain whether the foci or any other points enjoy a property like that which we have been considering, with respect to any pair of parallel lines other than conjugate ordinates. Refer the conic to any rectangular system with the given parallels as lines of reference. Denote by a the distance from the base of a point possessing the property in question, by b its distance from the S line, by L the distance between the parallels, and by m an indeterminate constant; then the equation of the conic is

$$(s - a)^2 + b^2 = m\,[(t - a)^2 + (b + L)^2], \text{ or}$$

$$s^2 - mt^2 - 2as + 2mat + a^2 + b^2 - m\,[a^2 + (b + L)^2] = 0.$$

Hence we have, with the usual meanings for A_{13} &c.,

$$A_{13} = -ma,\; A_{23} = -ma,\; A_{33} = -m,\; \frac{A_{13}}{A_{33}} = \frac{A_{23}}{A_{33}} = a;$$

and it follows at once (see p. 158, l. 26) that the parallels are perpendicular to an axis and that the point or points sought lie on the axis; and the absence of the term containing st shows that the parallels cut the axis in conjugate basics. We are thus brought back to the case already investigated; and conjugate ordinates are therefore the only pairs of parallels, and the foci the only points, enjoying the property in question. That it cannot belong to lines not parallel is evident.

The Conic referred to its Foci.—The distances of the extremities of the axis from the S initial are (p. 160) $l\dfrac{-c^2 \pm \sqrt{-c^2d^2}}{c^2+d^2}$, so that the distance of the centre from the S initial is $\dfrac{-lc^2}{c^2+d^2}$; and in like manner the distance of the centre from the T initial is $\dfrac{ld^2}{c^2+d^2}$. When the S initial is a focus, these distances are $\dfrac{e^2l}{1-e^2}$, $\dfrac{l}{1-e^2}$; so that the distances of the other focus from S and T are $\dfrac{2e^2l}{1-e^2}$ and $\dfrac{1+e^2}{1-e^2}l$. To transfer (see p. 152) the initials to the foci, therefore, we leave s unchanged and replace t by $\dfrac{lt-\dfrac{1+e^2}{1-e^2}ls}{\dfrac{2e^2l}{1-e^2}} = \dfrac{(1-e^2)t-(1+e^2)s}{2e^2}$, and the equation $s^2 = e^2(t^2+l^2)$ becomes $4e^2s^2 = (1-e^2)^2t^2 + (1+e^2)^2s^2 - 2(1-e^4)st + 4e^4l^2$ or $s^2+t^2-2\dfrac{1+e^2}{1-e^2}st + \dfrac{4e^4l^2}{(1-e^2)^2} = 0$. The last term in this equation is the square of the distance between the foci; denoting that distance by $2k$, and $\dfrac{1+e^2}{1-e^2}$ by h, the equation becomes

$$s^2+t^2-2hst+4k^2=0.$$

The equation of any other conic having the same foci would be $s^2+t^2-2h'st+4k^2=0$; combining this with the preceding equation, we have $st=0$ for the common tangents of the two curves. This condition is satisfied only by lines passing through the foci; and the equation of the curve shows that tangents passing through the foci are imaginary. Hence, all confocal conics have in common four imaginary tangents passing through the foci.

Denoting by σ, τ, the lengths of the perpendiculars dropped from S, T, upon the line s, t, and by ρ the cosine of the angle made by the line with the

base, we have $s = \frac{\sigma}{\rho}$, $t = \frac{\tau}{\rho}$, $\rho^2 = \frac{L^2 - (\sigma - \tau)^2}{L^2}$. By means of these equations we can transform any equation in s, t, in a rectangular system, into an equation in σ, τ; and the equation of the conic referred to its foci assumes, when thus transformed, a very simple form. We have, first, $\sigma^2 + \tau^2 - 2h\sigma\tau + 4k^2\rho^2 = 0$; but $\rho^2 = \frac{4k^2 - (\sigma - \tau)^2}{4k^2}$, and the equation reduces to $2(1 - h)\sigma\tau + 4k^2 = 0$, or $\sigma\tau = \frac{2k^2}{h - 1} = \frac{e^2 l^2}{1 - e^2} = b^2$, where b is half the minor axis (see p. 160, l. 8). Hence, the product of the perpendiculars dropped from the foci upon any tangent is equal to the square of half the minor axis.

To remove T to the centre, we have simply to replace t by $2t - s$ in the equation $s^2 + t^2 - 2hst + 4k^2 = 0$; so that we have for the equation of the curve referred to a focus and the centre, $\frac{1 + h}{2} s^2 + t^2 - (1 + h)\, st + k^2 = 0$; from which we obtain

$$\frac{1 + h}{2} \sigma^2 + \tau^2 - (1 + h)\, \sigma\tau + k^2\rho^2 = 0, \text{ or } \tau^2 + k^2\rho^2 = \frac{1 + h}{2} \sigma (2\tau - \sigma).$$

Now $\tau^2 + k^2\rho^2$ is the square of the distance from the centre to the foot of the perpendicular dropped from the focus, and $\sigma(2\tau - \sigma)$ is the product of the perpendiculars from the two foci, which is constant and equal to b^2: hence the distance from the centre to the foot of the perpendicular dropped from the focus upon any tangent is constant and equal to $\sqrt{\frac{1 + h}{2} b^2} = \sqrt{\frac{1}{1 - e^2} b^2} = a$; *i. e.*, the feet of all these perpendiculars are on a circle described upon the major axis as diameter.

The Conic referred to the Extremities of a Diameter. — Taking up the general equation of a conic referred to conjugate basics, $\frac{s^2}{c^2} + \frac{t^2}{d^2} = l^2$, and making the general substitution for a motion of the initials along the base, $s = \frac{mt' - m's'}{\lambda}$, $t = \frac{nt' - n's'}{\lambda}$ (where λ is the distance between the new initials), we obtain, dropping the accents of the variables,

$$\left(\frac{m'^2}{c^2} + \frac{n'^2}{d^2}\right) s^2 + \left(\frac{m^2}{c^2} + \frac{n^2}{d^2}\right) t^2 - 2\left(\frac{mm'}{c^2} + \frac{nn'}{d^2}\right) st = l^2\lambda^2.$$

If the extremities of the diameter are taken as the new initials, we have (p. 160) $\frac{m'^2}{c^2} + \frac{n'^2}{d^2} = 0$, $\frac{m^2}{c^2} + \frac{n^2}{d^2} = 0$; also, (since m, m', are the roots of the equation $(c^2 + d^2)\mu^2 + 2c^2l\mu + c^2l^2 = 0$, and n, n', the roots of the equation $(c^2 + d^2)\nu^2 - 2d^2l\nu + d^2l^2 = 0$) $mm' = \frac{c^2l^2}{c^2 + d^2}$, $nn' = \frac{d^2l^2}{c^2 + d^2}$, so that the equation of the curve referred to the extremities of a diameter is

$$-\frac{4st}{c^2 + d^2} = \lambda^2 = 4a^2; \text{ or } \left(\text{since } c^2 + d^2 = -\frac{b^2}{a^2}\right), \; st = b^2;$$

that is, the product of the distances cut off by any tangent on the tangents at the extremities of a diameter is equal to the square of half the conjugate diameter.

The equation of a conic in point-coordinates has for its coefficients the minors of the determinant of the coefficients of its equation in line-coordinates; hence the equation in point-coordinates of the conic referred to the extremities of a diameter is $b^2pq - \frac{1}{4} = 0$, or $pq = \frac{1}{4b^2}$. Representing, then, by π, $\varkappa$, the negative reciprocals of p, q, we have $\pi\varkappa = 4b^2$; that is, the lines joining any point of the conic with the extremities of a diameter cut off, on the tangents at those extremities, distances whose product is equal to the square of the conjugate diameter.

DESIDERATA AND SUGGESTIONS.

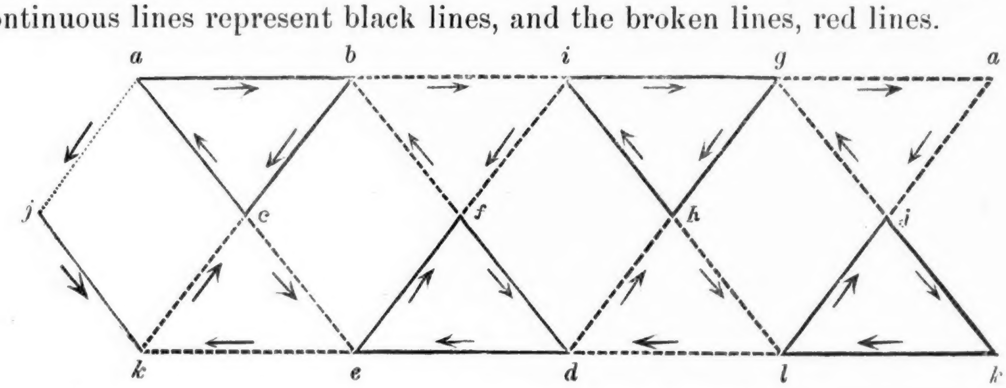
By Professor Cayley, *Cambridge, England.*

No. 2.—THE THEORY OF GROUPS: GRAPHICAL REPRESENTATION.

In regard to a substitution-group of the order n upon the same number of letters, I omitted to mention the important theorem that every substitution is *regular* (that is, either cyclical or composed of a number of cycles each of them of the same order). Thus in the group of 6 given in No. 1, writing a, b, c, d, e, f in place of $1, \alpha, \beta, \gamma, \delta, \varepsilon$, the substitutions of the group are 1, $ace.bfd$, $aec.bdf$, $ab.cd.ef$, $ad.be.cf$, $af.bc.de$.

Let the letters be represented by points; a change a into b will be represented by a directed line (line with an arrow) joining the two points; and therefore a cycle abc, that is, a into b, b into c, c into a, by the three sides of the trilateral abc, with the three arrows pointing accordingly, and similarly for the cycles $abcd$, &c.: the cycle ab means a into b, b into a, and we have here the line ab with a two headed arrow pointing both ways; such a line may be regarded as a bilateral. A substitution is thus represented by a multilateral or system of multilaterals, each side with its arrow; and in the case of a regular substitution the multilaterals (if more than one) have each of them the same number of sides. To represent two or more substitutions we require different colours, the multilaterals belonging to any one substitution being of the same colour.

In order to represent a group we need to represent only independent substitutions thereof; that is, substitutions such that no one of them can be obtained from the others by compounding them together in any manner. I take as an example a group of the order 12 upon 12 letters, where the number of independent substitutions is $= 2$. See the diagram, wherein the continuous lines represent black lines, and the broken lines, red lines.



The diagram is drawn, in the first instance, with the arrows but without the letters, which are then affixed *at pleasure;* viz: the *form of group* is quite independent of the way in which this is done, though the group itself is of course dependent upon it. The diagram shows two substitutions, each of them of the third order, one represented by the black triangles, and the other by the red triangles. It will be observed that there is *from* each point of the diagram (that is, in the direction of the arrow) one and only one black line, and one and only one red line; hence, a symbol, B, "move along a black line," B^2, move successively along two black lines," BR (read always from right to left), "move first along a red line and then along a black line," has in every case a perfectly definite meaning and determines the path when the initial point is given; any such symbol may be spoken of as a "route."

The diagram has a remarkable property, *in virtue whereof it in fact represents a group.* It may be seen that any route leading from some one point a to itself, leads also from every other point to itself, or say from b to b, from c to c, ... and from l to l. We hence see that a route applied in succession to the whole series of initial points or letters *abcdefghijkl*, gives a new arrangement of these letters, wherein no one of them occupies its original place; a route is thus, in effect, a substitution. Moreover, we may regard as distinct routes, those which lead from a to a, to b, to c, . . . to l, respectively. We have thus 12 substitutions (the first of them, which leaves the arrangement unaltered, being the substitution unity), and these 12 substitutions form a group. I omit the details of the proof; it will be sufficient to give the square obtained by means of the several routes, or substitutions, performed upon the primitive arrangement *abcdefghijkl*, and the cyclical expressions of the

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*
b	*c*	*a*	*e*	*f*	*d*	*h*	*i*	*g*	*k*	*l*	*j*
c	*a*	*b*	*f*	*d*	*e*	*i*	*g*	*h*	*l*	*j*	*k*
d	*l*	*h*	*a*	*g*	*j*	*e*	*c*	*k*	*f*	*i*	*b*
e	*j*	*i*	*b*	*h*	*k*	*f*	*a*	*l*	*d*	*g*	*c*
f	*k*	*g*	*c*	*i*	*l*	*d*	*b*	*j*	*e*	*h*	*a*
g	*f*	*k*	*l*	*c*	*i*	*j*	*d*	*b*	*a*	*e*	*h*
h	*d*	*l*	*j*	*a*	*g*	*k*	*e*	*c*	*b*	*f*	*i*
i	*e*	*j*	*k*	*b*	*h*	*l*	*f*	*a*	*c*	*d*	*g*
j	*i*	*e*	*h*	*k*	*b*	*a*	*l*	*f*	*g*	*c*	*d*
k	*g*	*f*	*i*	*l*	*c*	*b*	*j*	*d*	*h*	*a*	*e*
l	*h*	*d*	*g*	*j*	*a*	*c*	*k*	*e*	*i*	*b*	*f*

1
abc . *def* . *ghi* . *jkl* (= B)
acb . *dfe* . *gih* . *jlk*
ad . *bl* . *ch* . *eg* . *fj* . *ik*
aeh . *bjd* . *cil* . *fkg*
afl . *bkh* . *cgd* . *eij*
agj . *bfi* . *cke* . *dlh*
ahe . *bdj* . *cli* . *fgk*
ai . *be* . *cj* . *dk* . *fh* . *gl*
ajg . *bif* . *cek* . *dhl* (= R).
ak . *bg* . *cf* . *di* . *el* . *hj*
alf . *bhk* . *cdg* . *eji*

substitutions themselves: it will be observed that the substitutions are unity, 3 substitutions of the order (or index) 2, and 8 substitutions of the order (or index) 3.

It may be remarked that the group of 12 is really the group of the 12 positive substitutions upon 4 letters *abcd*, viz., these are 1, *abc*, *acb*, *abd*, *adb*, *acd*, *adc*, *bcd*, *bdc*, *ab.cd*, *ac.bd*, *ad.bc*.

CAMBRIDGE, *16th May, 1878.*

ON THE ELASTIC POTENTIAL OF CRYSTALS.

By William E. Story.

The theory of elasticity in its applications to homogeneous isotropic bodies has been more or less completely developed by Poisson, Cauchy, Lamé and Clebsch, while Riemann's "Differential-gleichungen" contains an admirable abstract of it.

On a particle of a solid body may act three kinds of forces:

1) *External forces*, attractions and repulsions by external masses;

2) *Surface-forces*, pressures or tensions applied directly to elements of the surface;

3) *Elastic forces*, molecular forces due to the influence of neighboring particles of the body itself.

Elastic forces are produced by unequal displacements of the particles, causing a change in their relative positions, and hence a change in the molecular forces acting between them. Surface-forces affect the internal particles of the body only indirectly, by causing a displacement of the particles of the surface, giving rise to elastic forces which spread through the interior. It is the elastic forces alone that I shall consider in this paper. These may be considered as forces tending to prevent any change in the relative positions of the particles on one side of a section (plane or curved) of the body with respect to those on the other side; and, because the mutual action of the particles becomes inappreciable when the distance between them exceeds a very small limit, a particle on one side of the section will be affected only by those particles on the other side which lie very near the section and within a small distance of the particle in question. So that the whole mutual effect of the particles on the two sides of the section may be treated as the sum of the mutual effects of particles on the two sides of the very small portions (which may be considered plane-elements) of the section; and it is only necessary to consider the elastic forces on such plane-elements, which vary from element to element of the section, but, in the immediate vicinity of any given point of the section, are proportional to the plane-elements on which they act. It is there-

fore convenient to call the *components* of the elastic force acting on any plane-element, the ratios of the actual components to the area of the element. The components of the elastic force acting on any plane-element passing through a given point may be expressed in terms of the components of the elastic forces acting on any three mutually perpendicular plane-elements through the point, *e. g.* three elements parallel to the planes of a rectangular system of coordinates.

Let x, y, z be the coordinates of any point of the body referred to a fixed rectangular system, and let three plane-elements of areas ε_x, ε_y, ε_z, whose normals are parallel to the axes of x, y, z respectively, pass through this point; then in notation of Cauchy the components of the elastic forces acting on ε_x, ε_y, ε_z in the directions of the three axes will be X_x, Y_x, Z_x; X_y, Y_y, Z_y; X_z, Y_z, Z_z: in each of which the large letter denotes the direction of the component, and the suffix the direction of the normal to the plane-element on which it acts. Let, further, the density at the point x, y, z be ρ.

If x, y, z is a point in the interior of the body, let the components of the resultant of the external forces acting on a very small mass m at this point be mX, mY, mZ; then the conditions for the equilibrium of this small mass are

$$(1)\quad \left\{\begin{aligned} &\rho X + \frac{\delta X_x}{\delta x} + \frac{\delta X_y}{\delta y} + \frac{\delta X_z}{\delta z} = 0, \\ &\rho Y + \frac{\delta Y_x}{\delta x} + \frac{\delta Y_y}{\delta y} + \frac{\delta Y_z}{\delta z} = 0, \\ &\rho Z + \frac{\delta Z_x}{\delta x} + \frac{\delta Z_y}{\delta y} + \frac{\delta Z_z}{\delta z} = 0, \\ &Y_z = Z_y\,,\ Z_x = X_z\,,\ X_y = Y_x\,. \end{aligned}\right.$$

If, however, x, y, z is a point of the surface, let α, β, γ be the direction-cosines of the inner normal to the surface, and $\varepsilon\Xi$, εH, εZ the components of the surface-force on an element ε of the surface; then the conditions for the equilibrium of an infinitesimal solid element immediately under the surface-element ε are

$$(2)\quad \left\{\begin{aligned} &\Xi + \alpha X_x + \beta X_y + \gamma X_z = 0, \\ &H + \alpha Y_x + \beta Y_y + \gamma Y_z = 0, \\ &Z + \alpha Z_x + \beta Z_y + \gamma Z_z = 0, \end{aligned}\right.$$

where the surface-force is considered positive if a pressure and negative if a tension.

It is well known that, if u, v, w are the displacements, in the directions of the axes, of the point x, y, z, and if

$$(3)\quad \begin{cases} \dfrac{\delta u}{\delta x} = x_x\,, \dfrac{\delta v}{\delta y} = y_y\,, \dfrac{\delta w}{\delta z} = z_z\,, \\ \dfrac{\delta v}{\delta z} + \dfrac{\delta w}{\delta y} = y_z = z_y\,, \dfrac{\delta w}{\delta x} + \dfrac{\delta u}{\delta z} = z_x = x_z\,, \dfrac{\delta u}{\delta y} + \dfrac{\delta v}{\delta x} = x_y = y_x\,, \end{cases}$$

then there exists a function Φ of x_x, y_y, z_z, y_z, z_x, x_y, such that

$$(4)\quad \begin{cases} X_x = \dfrac{\delta\Phi}{\delta x_x},\ Y_y = \dfrac{\delta\Phi}{\delta y_y},\ Z_z = \dfrac{\delta\Phi}{\delta z_z}, \\ Y_z = Z_y = \dfrac{\delta\Phi}{\delta y_z},\ Z_x = X_z = \dfrac{\delta\Phi}{\delta z_x},\ X_y = Y_x = \dfrac{\delta\Phi}{\delta x_y}. \end{cases}$$

This function Φ, which may be called the *elastic potential,* is homogeneous of the second degree in x_x, y_y, z_z, y_z, z_x, x_y, and contains therefore in general 21 terms, whose coefficients are characteristic for the body under consideration, depending only on its structure in different directions. For a homogeneous body, *i. e.* one, every particle of which is similarly surrounded by particles, the coefficients of Φ are constant, but for a non-homogeneous body they differ from point to point, *i. e,* are functions of the coordinates x, y, z. For a homogeneous isotropic body the form of Φ has been determined to be

$$(5)\qquad \Phi = \tfrac{1}{2}\lambda\,(x_x + y_y + z_z)^2 + \mu\,(x_x^2 + y_y^2 + z_z^2) + \tfrac{1}{2}\mu\,(y_z^2 + z_x^2 + x_y^2)\,,$$

where λ and μ are constants. I now propose to determine its form for homoneous crystalline bodies.

Each crystalline system has a form of symmetry with respect to three mutually perpendicular axes which is not altered by a transformation to any one of certain other systems of mutually perpendicular axes having the same origin. We may then take one of these systems of axes as the basis of a system of coordinates x, y, z, and any other of them as the basis of a system of coordinates x', y', z'. The external form of symmetry of a crystal must, it would seem, be also a form of symmetry of arrangement of its particles; if this be so, a transformation of coordinates from x, y, z to x', y', z' must leave the coefficients unchanged. This will be the basis of the following determinations of the relations between the coefficients, on which relations the form of Φ depends.

The transformation from x, y, z to x', y', z' can be effected by a combination of two simple kinds of transformations, viz. by inversions of axes and

rotations about axes through certain angles. There are six such transformations, which I will designate by the first six letters of the alphabet, as follows:
a) inversion of the axis of x,

$$x' = -x\,,\ y' = y\,,\ z' = z\,,\ u = -u'\,,\ v = v'\,,\ w = w'\,,$$
$$x_x = x'_{x'}\,,\ y_y = y'_{y'}\,,\ z_z = z'_{z'}\,,\ y_z = y'_{z'}\,,\ z_x = -z'_{x'}\,,\ x_y = -x'_{y'}\,;$$

b) inversion of the axis of y,

$$x' = x\,,\ y' = -y\,,\ z' = z\,,\ u = u'\,,\ v = -v'\,,\ w = w'\,,$$
$$x_x = x'_{x'}\,,\ y_y = y'_{y'}\,,\ z_z = z'_{z'}\,,\ y_z = -y'_{z'}\,,\ z_x = z'_{x'}\,,\ x_y = -x'_{y'}\,;$$

c) inversion of the axis of z,

$$x' = x\,,\ y' = y\,,\ z' = -z\,,\ u = u'\,,\ v = v'\,,\ w = -w'\,,$$
$$x_x = x'_{x'}\,,\ y_y = y'_{y'}\,,\ z_z = z'_{z'}\,,\ y_z = -y'_{z'}\,,\ z_x = -z'_{x'}\,,\ x_y = x'_{y'}\,;$$

d) rotation about the axis of x through the angle α,

$$x' = x\,,\ y' = y\cos\alpha + z\sin\alpha\,,\ z' = -y\sin\alpha + z\cos\alpha\,,$$
$$u = u'\,,\ v = v'\cos\alpha - w'\sin\alpha\,,\ w = v'\sin\alpha + w'\cos\alpha\,,$$
$$x_x = x'_{x'}\,,\ y_y = y'_{y'}\cos^2\alpha - y'_{z'}\sin\alpha\cos\alpha + z'_{z'}\sin^2\alpha\,,$$
$$z_z = y'_{y'}\sin^2\alpha + y'_{z'}\sin\alpha\cos\alpha + z'_{z'}\cos^2\alpha\,,$$
$$y_z = (y'_{y'} - z'_{z'})\sin 2\alpha + y'_{z'}\cos 2\alpha\,,$$
$$z_x = x'_{y'}\sin\alpha + z'_{x'}\cos\alpha\,,\ x_y = x'_{y'}\cos\alpha - z'_{x'}\sin\alpha\,;$$

e) rotation about the axis of y through the angle β, by formulæ found from those of *d*) by changing x to y, y to z, z to x, and α to β;
f) rotation about the axis of z through the angle γ, by formulæ found from *d*) by changing x to z, y to x, z to y, and α to γ.

The general form of Φ is

$$(6)\quad \left\{\begin{aligned} \Phi = a_{11} + x_x^2 + a_{22}y_y^2 + a_{33}z_z^2 + 2a_{23}y_yz_z + 2a_{13}x_xz_z + 2a_{12}x_xy_y \\ + 2a_{14}x_xy_z + 2a_{15}x_xz_x + 2a_{16}x_xx_y + 2a_{24}y_yy_z + 2a_{25}y_yz_x \\ + 2a_{26}y_yx_y + 2a_{34}z_zy_z + 2a_{35}z_zz_x + 2a_{36}z_zx_y + a_{44}y_z^2 \\ + a_{55}z_x^2 + a_{66}x_y^2 + 2a_{56}z_xx_y + 2a_{46}x_yy_z + 2a_{45}y_zz_x\,. \end{aligned}\right.$$

That the transformation *a*) shall leave Φ unaltered in form it is necessary and sufficient that

$$(7)\qquad a_{15} = a_{25} = a_{35} = a_{45} = a_{56} = a_{16} = a_{26} = a_{36} = a_{46} = 0.$$

That the transformation *b*) shall leave the form of Φ unaltered it is necessary and sufficient that

$$(8)\qquad a_{14} = a_{24} = a_{34} = a_{45} = a_{16} = a_{26} = a_{36} = a_{56} = 0.$$

That the transformation *c*) shall leave the form of Φ unaltered it is necessary and sufficient that

$$(9)\qquad a_{14} = a_{24} = a_{34} = a_{46} = a_{15} = a_{25} = a_{35} = a_{56} = 0.$$

That the transformation *d*) shall leave the form of Φ unaltered it is necessary and sufficient that

$$(10)\left\{\begin{array}{l}
(a_{12}-a_{13})\sin\alpha=0\,,\ (a_{55}-a_{66})\sin\alpha=0\,,\ (a_{22}-a_{33})\sin\alpha=0\,,\\
a_{14}\sin\alpha=0\,,\ a_{56}\sin\alpha=0\,,\ a_{15}(1-\cos\alpha)=0\,,\ a_{16}(1-\cos\alpha)=0\,,\\
(a_{22}+a_{33}-2a_{23}-4a_{44})\sin\alpha\cos\alpha=0\,,\\
(a_{24}+a_{34})\sin\alpha=0\,,\ (a_{24}-a_{34})\sin\alpha\cos\alpha=0\,,\\
(a_{25}+a_{35})(1-\cos\alpha)=0\,,\ (a_{26}+a_{36})(1-\cos\alpha)=0\,,\\
[(a_{25}-a_{35})+(a_{26}-a_{36})]\sin\alpha\cos\alpha+(\cos\alpha-\cos2\alpha-\sin\alpha)\,a_{45}\\
+(\cos\alpha-\cos2\alpha+\sin\alpha)\,a_{46}=0\,,\\
[(a_{25}-a_{35})-(a_{26}-a_{36})]\sin\alpha\cos\alpha+(\cos\alpha-\cos2\alpha+\sin\alpha)\,a_{45}\\
-(\cos\alpha-\cos2\alpha-\sin\alpha)\,a_{46}=0\,,\\
(a_{25}-a_{35})(\cos\alpha-\cos2\alpha-\sin\alpha)+(a_{26}-a_{36})(\cos\alpha-\cos2\alpha+\sin\alpha)\\
-4\,(a_{45}+a_{46})\sin\alpha\cos\alpha=0\,,\\
(a_{25}-a_{35})(\cos\alpha-\cos2\alpha+\sin\alpha)-(a_{26}-a_{36})(\cos\alpha-\cos2\alpha-\sin\alpha)\\
-4\,(a_{45}-a_{46})\sin\alpha\cos\alpha=0\,.
\end{array}\right.$$

The conditions (10) are, of course, satisfied by $\alpha=0$. They will also be satisfied by a multiple of 90° when certain relations hold between the coefficients, as follows:

if $\alpha=90°$ or $-90°$,

$$(11)\left\{\begin{array}{l}
a_{14}=a_{15}=a_{16}=a_{25}=a_{26}=a_{35}=a_{36}=a_{45}=a_{46}=a_{56}=0\,,\\
a_{12}=a_{13}\,,\ a_{22}=a_{33}\,,\ a_{55}=a_{66}\,,\ a_{24}=-a_{34}\,;
\end{array}\right.$$

if $\alpha=180°$,

$$(12)\qquad a_{15}=a_{25}=a_{35}=a_{45}=a_{16}=a_{26}=a_{36}=a_{46}=0\,,$$

which are identical with (7), so that, if an axis can be inverted without producing any effect on Φ, a rotation about that axis through an angle of 180° will also produce no effect, and *vice versa*, as is also evident from the fact that all three axes may be inverted without producing any effect on the most general form (6) of Φ.

If α has neither of the above values,

$$(13)\left\{\begin{array}{l}
a_{14}=a_{24}=a_{34}=a_{15}=a_{16}=a_{56}=0\,,\\
a_{12}=a_{13}\,,\ a_{22}=a_{33}=a_{23}+2a_{44}\,,\ a_{55}=a_{66}\,,\ a_{25}=-a_{35}\,,\ a_{26}=-a_{36}\,,
\end{array}\right.$$

and either

$$(13_1)\qquad a_{25}=a_{35}=a_{45}=a_{26}=a_{36}=a_{46}=0\,,$$

or

$$(13_2)\qquad 4\cos^6\alpha+4\cos^5\alpha-2\cos^4\alpha-2\cos^3\alpha-4\cos^2\alpha-4\cos\alpha-1=0$$

(whose only real roots between — 1 and + 1 are $\cos \alpha = - 0.607 +$ and $\cos \alpha = - 0.389 -$), *i e.* if a rotation about an axis through any other angle than 90°, 180°, 270° (or — 90°), $\cos^{[-1]}(-0.607 +)$ or $\cos^{[-1]}(-0.389 -)$ does not affect the form of Φ, a rotation about this axis through any angle whatever will not affect it, *i. e.* there exists perfect symmetry about this angle.

The transformations *e*) and *f*) give conditions similar to (10), (11), (12) and (13), found from them by a cyclic interchange in each set of suffixes 1, 2, 3 and 4, 5, 6 and in α, β, γ.

Each of the crystalline systems admits of a characteristic combination of the above transformations.

The ***triclinic*** system admits only of an inversion of all the axes at the same time, which leaves the most general form of Φ unchanged without any conditions between the coefficients, *i. e.* (6) is the general form of Φ for triclinic crystals.

The ***monoclinic*** system, in which the axis of x is taken in that axis of the crystal which is perpendicular to the plane of the two others, allows only the transformation *a*) and *d*) in which $\alpha = 180°$, *i. e.* its conditions are (7) and (12), which are identical, and

$$(14) \left\{ \begin{aligned} \Phi = {} & a_{11}x_x^2 + a_{22}y_y^2 + a_{33}z_z^2 + 2a_{23}y_yz_z + 2a_{13}x_xz_z \\ & + 2a_{12}x_xy_y + 2a_{14}x_xy_z + 2a_{24}y_yy_z + 2a_{34}z_zy_z + a_{44}y_z^2 \\ & + a_{55}z_x^2 + a_{66}x_y^2 + 2a_{56}z_xx_y \,. \end{aligned} \right.$$

The ***trimetric*** system, in which the axes of x, y and z coincide with the axes of the crystal, admits of each of the transformations *a*), *b*) and *c*), hence for this system the conditions are (7), (8) and (9), and the form of Φ is

$$(15) \left\{ \begin{aligned} \Phi & = a_{11}x_z^2 + a_{22}y_y^2 + a_{33}z_z^2 + 2a_{23}y_yz_z + 2a_{13}x_xz_z + 2a_{12}x_xy_y \\ & = a_{44}y_2^z + a_{55}z_x^2 + a_{66}x_y^2. \end{aligned} \right.$$

The ***dimetric*** system, whose principal axis is the axis of x and whose secondary axes are those of y and z, admits of the transformations *a*), *b*), *c*) and *d*) where $\alpha = 90°$, so that the conditions are (7), (8), (9) and (11), and the form of Φ is

$$(16) \left\{ \begin{aligned} \Phi = {} & a_{11}x_x^2 + a_{22}\,(y_y^2 + z_z^2) + 2a_{23}y_yz_z + 2a_{12}x_x\,(y_y + z_z) \\ & + a_{44}y_z^2 + a_{55}\,(z_x^2 + x_y^2). \end{aligned} \right.$$

The ***monometric*** system, whose axes are the axes of x, y and z, admits of the transformations *a*), *b*), *c*), *d*), *e*) and *f*) for $\alpha = 90°$, $\beta = 90°$, $\gamma = 90°$; so that the conditions are (7), (8), (9), (11) and, similar to (11),

$$a_{14} = a_{16} = a_{24} = a_{25} = a_{26} = a_{34} = a_{36} = a_{56} = a_{46} = a_{45} = 0\,,$$
$$a_{23} = a_{12}\,,\ a_{11} = a_{33}\,,\ a_{44} = a_{66}\,,\ a_{15} = -\,a_{35}\,,$$
$$a_{14} = a_{15} = a_{24} = a_{25} = a_{34} = a_{35} = a_{36} = a_{56} = a_{46} = a_{45} = 0\,,$$
$$a_{23} = a_{13}\,,\ a_{11} = a_{22}\,,\ a_{44} = a_{55}\,,\ a_{16} = -\,a_{26}\,,$$

and the form of Φ is

$$(17)\quad \Phi = a_{11}\,(x_x^2 + y_y^2 + z_z^2) + 2a_{12}\,(y_y z_z + z_z x_x + x_x y_y) + a_{44}\,(y_z^2 + z_x^2 + x_y^2).$$

The *hexagonal*, as well as *any regular prismatic* system, other than one whose secondary axes include an angle of 90° (because the conditions (13_1) omitted in case $\alpha = \cos^{[-1]}(-0.607+)$ or $\alpha = \cos^{[-1]}(-0.389-)$ are more than replaced by (7)), whose principal axis is that of x, admits of the transformations a), b), c), d); the conditions are then (7), (8), (9), (13) and (13_1), and Φ has the form

$$(18)\quad \left\{ \begin{aligned} \Phi = a_{11}x_x^2 + a_{22}\,(y_y^2 + z_z^2) + 2a_{23}y_y z_z + 2a_{12}x_x\,(y_y + z_z) \\ + \tfrac{1}{2}\,(a_{22} - a_{23})\,y_z^2 + a_{55}\,(z_x^2 + x_y^2)\,. \end{aligned} \right.$$

It is to be remarked that (16) differs from (18) only in the coefficient of y_z^2, which is independent of the other coefficients in the former case, but dependent on the coefficients of $y_y^2 + z_z^2$ and $y_y\, z_z$ in the latter case, so that, as far as their elastic relations are concerned, square prismatic crystals seem to be the most complex of regular prismatic forms, in as much as they contain one more constant than any other.

From the above determinations of Φ it will be seen that the elastic potentials of isotropic bodies, of monometric, hexagonal, dimetric, trimetric, monoclinic and triclinic crystals contain respectively 2, 3, 5, 6, 9, 13 and 21 constants. The forms of Φ, (5), (17), (18), (16), (15_1) (14) and (6) substituted in (4) give the forms of the components of the elastic force in these different cases of homogeneity, and these components substituted in (1) and (2) give the conditions for equilibrium under a given set of external forces X, Y, Z and surface-forces Ξ, H, Z. The conditions for motion, *e. g.* elastic vibrations, will be found from those for equilibrium by the substitution of $X - \dfrac{d^2u}{dt^2}$, $Y - \dfrac{d^2v}{dt^2}$, $Z - \dfrac{d^2w}{dt^2}$ for X, Y, Z respectively, t denoting time.

THÉORIE DES FONCTIONS NUMÉRIQUES SIMPLEMENT PÉRIODIQUES.

PAR EDOUARD LUCAS, *Professeur au Lycée Charlemagne, Paris.*

CE mémoire a pour objet l'étude des fonctions symétriques des racines d'une équation du second degré, et son application à la théorie des nombres premiers. Nous indiquons dès le commencement, l'analogie complète de ces fonctions symétriques avec les fonctions circulaires et hyperboliques; nous montrons ensuite la liaison qui existe entre ces fonctions symétriques et les théories des déterminants, des combinaisons, des fractions continues, de la divisibilité, des diviseurs quadratiques, des radicaux continus, de la division de la circonférence, de l'analyse indéterminée du second degré, des résidus quadratiques, de la décomposition des grands nombres en facteurs premiers, etc. Cette méthode est le point de départ d'une étude plus compléte, des propriétés des fonctions symétriques des racines d'une équation algébrique, de degré quelconque, à coefficients commensurables, dans leurs rapports avec les théories des fonctions elliptiques et abéliennes, des résidus potentiels, et de l'analyse indeterminee des degrés supérieurs.

SECTION I.

Définition des fonctions numériques simplement périodiques.

Désignons par a et b les deux racines de l'équation

$$(1) \qquad x^2 = Px - Q\,,$$

dont les coefficients P et Q sont des nombres entiers, positifs ou négatifs, et premiers entre eux. On a

$$a + b = P\,, \qquad ab = Q\,;$$

et, en désignant par δ la différence $a - b$ des racines, et par Δ le carré de cette différence, on a encore

$$a = \frac{P+\delta}{2}\,, \qquad b = \frac{P-\delta}{2}\,, \qquad \delta = \sqrt{\Delta} = \sqrt{P^2 - 4\,Q}.$$

Cela posé, nous considérerons les deux fonctions numériques U et V définies par les égalités

$$(2) \qquad U_n = \frac{a^n - b^n}{a - b}, \qquad V_n = a^n + b^n.$$

Ces fonctions U_n et V_n donnent naissance, pour toutes les valeurs entières et positives de n, à trois séries d'espèces différentes, selon la nature des racines a et b de l'équation (1). Cette équation peut avoir :

1°. Les racines réelles et entières ;

2°. Les racines réelles et incommensurables ;

3°. Les racines imaginaires.

Les *fonctions numériques de première espèce* correspondent à toutes les valeurs entières de a et de b, et peuvent être calculées directement, pour toutes les valeurs entières et positives de n, par l'emploi des formules (2). Si l'on suppose plus particulièrement $a = 2$ et $b = 1$, on trouve, en formant les valeurs de U_n et de V_n, les séries récurrentes

n:	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11, ... ,
U_n:	0,	1,	3,	7,	15,	31,	63,	127,	255,	511,	1023,	2047, ... ,
V_n:	2,	3,	5,	9,	17,	33,	65,	129,	257,	513,	1025,	2049, ...

étudiées pour la première fois par l'illustre FERMAT. Nous observerons, dès maintenant, que la série des V_n est contenue, pour les trois cas que nous considérons, dans la série des U_n, puisque les formules (2) nous donnent la relation générale

$$(3) \qquad U_{2n} = U_n V_n.$$

Les *fonctions numériques de seconde espèce* correspondent à toutes les valeurs incommensurables de a et de b dont la somme et le produit sont commensurables. On peut les calculer en fonction de la somme P et du discriminant Δ de l'équation proposée, au moyen des formules suivantes. Le développement du binôme nous donne

$$2^n a^n = P^n + \frac{n}{1} P^{n-1}\delta + \frac{n(n-1)}{1.2} P^{n-2}\delta^2 + \frac{n(n-1)(n-2)}{1.2.3} P^{n-3}\delta^3 + \ldots + \delta^n,$$

$$2^n b^n = P^n - \frac{n}{1} P^{n-1}\delta + \frac{n(n-1)}{1.2} P^{n-2}\delta^2 - \frac{n(n-1)(n-2)}{1.2.3} P^{n-3}\delta^3 - \ldots + (-\delta)^n;$$

et, par soustraction et par addition,

$$(4)\quad \left\{\begin{aligned} 2^{n-1}U_n &= \frac{n}{1}P^{n-1} + \frac{n(n-1)(n-2)}{1.2.3}P^{n-3}\Delta \\ &\qquad + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}P^{n-5}\Delta^2 + \ldots\ldots, \\ 2^{n-1}V_n &= P^n + \frac{n(n-1)}{1.2}P^{n-2}\Delta + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}P^{n-4}\Delta^2 + \ldots\ldots \end{aligned}\right.$$

On obtient ainsi, pour les premiers termes,

$$\begin{array}{lllll} U_0 = 0, & U_1 = 1, & U_2 = P, & U_3 = P^2 - Q, & U_4 = P^3 - 2PQ, \\ V_0 = 2, & V_1 = P, & V_2 = P^2 - 2Q, & V_3 = P^3 - 3PQ, & V_4 = P^4 - 4P^2Q + 2Q^2. \end{array}$$

Les fonctions numériques de seconde espèce les plus simples correspondent aux hypothèses

$$P = 1, \qquad Q = -1, \qquad \Delta = 5,$$

ou à l'équation
$$x^2 = x + 1;$$
on a, dans ce cas,

$$a = 2\sin\frac{3\pi}{10} = \frac{1+\sqrt{5}}{2}, \qquad b = -2\sin\frac{\pi}{10} = \frac{1-\sqrt{5}}{2},$$

et, par suite, en désignant par u_n et v_n les fonctions qui en résultent,

$$u_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}, \qquad v_n = \frac{(1+\sqrt{5})^n + (1-\sqrt{5})^n}{2^n}.$$

On forme ainsi, pour les premières valeurs de n entières et positives, les séries

n:	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11, ... ,
u_n:	0,	1,	1,	2,	3,	5,	8,	13,	21,	34,	55,	89, ... ,
v_n:	2,	1,	3,	4,	7,	11,	18,	29,	47,	76,	123,	199, ...

La série des u_n a été considérée pour la première fois par LÉONARD FIBONACCI, de Pise.* Elle a été étudiée par ALBERT GIRARD,† qui a observé que les trois nombres u_n, u_n, u_{n+1} forment un triangle isoscèle dont l'angle au sommet est à fort peu près égal à l'angle du pentagone régulier. ROBERT SIMSON ‡ a

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* *Il liber Abbaci di Leonardo Pisano, pubblicato secondo la lezione del Codice Magliabechiano, da* B. BONCOMPAGNI. Roma, 1867. Pag. 283 et 284.

† *L'arithmétique de* SIMON STEVIN, *de Bruges, revue, corrigée et augmentée de plusieurs traictez et annotations par* ALBERT GIRARD, etc. Leide, 1633. Pag. 169 et 170.

‡ *Philosophical Transactions of the Royal Society of London*, Vol. xlviii, Part I, for the year 1753. An explication of an obscure passage in Albert Girard's Commentary upon Simon Stevin's Works. Pag. 368 et suiv.

fait remarquer en 1753, que cette série est donnée par le calcul des quotients et des fractions convergentes des expressions irrationnelles

$$\frac{\sqrt{5}+1}{2} \quad \text{et} \quad \frac{\sqrt{5}-1}{2}.$$

En 1843, J. BINET * donne, au moyen de cette série, l'expression du dénombrement des combinaisons discontigues. En 1844, LAMÉ † indique l'application que l'on peut faire de cette série à la détermination d'une limite supérieure du nombre des operations à faire dans la recherche du plus grand commun diviseur de deux nombres entiers.

Nous prendrons aussi quelquefois pour exemple la série U_n de seconde espèce, donnée par les hypothèses

$$P = 2, \qquad Q = -1, \qquad \Delta = 2^2.2,$$

ou par l'équation

$$x^2 = 2x + 1.$$

On a alors les séries

n:	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11, ...
U_n:	0,	1,	2,	5,	12,	29,	70,	169,	408,	985,	2378,	5741, ...
V_n:	2,	2,	6,	14,	34,	82,	198,	478,	1154,	2786,	6726,	16238, ...

que nous désignerons sous le nom de SÉRIES DE PELL, en l'honneur du mathématicien de ce nom qui résolut, le premier, un célèbre problème d'analyse indéterminée proposé par FERMAT, et concernant la résolution en nombres entiers, de l'équation indéterminée

$$x^2 - \Delta y^2 = \pm 1.$$

Les *fonctions numériques de troisième espèce* correspondent à toutes les valeurs imaginaires de a et de b dont la somme et le produit sont réels et commensurables. Les plus simples proviennent des hypothèses

$$P = 1, \qquad Q = 1, \qquad \Delta = -3;$$

on a, dans ce cas,

$$a = \frac{1+\sqrt{-3}}{2}, \qquad b = \frac{1-\sqrt{-3}}{2},$$

par conséquent a et b sont les racines cubiques imaginaires de l'unité négative; de plus,

$$U_{3n} = 0, \qquad U_{3n+1} = (-1)^n, \qquad U_{3n+2} = (-1)^n.$$

* *Comptes rendus de l'académie des sciences de Paris*, tome, xvii, pag 562 ; tome xix, pag. 939.

† *Comptes rendus*, etc., tome xix, pag. 867.

Ainsi les valeurs de U_n reviennent périodiquement dans l'ordre

$$0,\ 1,\ 1,\ 0,\ -1,\ -1,\ldots\ldots$$

et donnent lieu à un grand nombre de formules simples déduites des propriétés générales des fonctions U_n et V_n, et concernant la trisection de la circonférence.

Quelquefois aussi nous considèrerons les séries analogues déduites de l'équation

$$x^2 = 2x - 2,$$

dans laquelle

$$a = 1 + \sqrt{-1}, \qquad b = 1 - \sqrt{-1}, \qquad \Delta = -2^2,$$

et les séries déduites de l'équation

$$x^2 = 2x - 3,$$

dans laquelle

$$a = 1 + \sqrt{-2}, \qquad b = 1 - \sqrt{-2}, \qquad \Delta = -2 \times 2^2;$$

nous désignerons les séries obtenues dans cette dernière hypothèse, sous le nom de *séries conjuguées* de PELL.

SECTION II.

Des relations des fonctions U_n et V_n avec les fonctions circulaires et hyperboliques.

Si l'on fait

$$z = \frac{n}{2} \text{ Log. nép. } \frac{a}{b},$$

dans les formules

$$\cos(z\sqrt{-1}) = \frac{e^z + e^{-z}}{2},$$

$$\sin(z\sqrt{-1}) = \frac{e^z - e^{-z}}{2\sqrt{-1}},$$

on obtient

$$\cos\left(\frac{n\sqrt{-1}}{2} \text{ Log. } \frac{a}{b}\right) = \frac{1}{2}\left[\frac{a^{\frac{n}{2}}}{b^{\frac{n}{2}}} + \frac{b^{\frac{n}{2}}}{a^{\frac{n}{2}}}\right],$$

$$\sin\left(\frac{n\sqrt{-1}}{2} \text{ Log. } \frac{a}{b}\right) = \frac{1}{2\sqrt{-1}}\left[\frac{a^{\frac{n}{2}}}{b^{\frac{n}{2}}} - \frac{b^{\frac{n}{2}}}{a^{\frac{n}{2}}}\right];$$

on a donc, entre les fonctions U_n et V_n, et les fonctions circulaires, les deux relations

$$(5)\quad \begin{cases} V_n = 2Q^{\frac{n}{2}} \cos\left(\dfrac{n\sqrt{-1}}{2} \operatorname{Log.} \dfrac{a}{b}\right), \\[2ex] U_n = \dfrac{2Q^{\frac{n}{2}}}{\sqrt{-\Delta}} \sin\left(\dfrac{n\sqrt{-1}}{2} \operatorname{Log.} \dfrac{a}{b}\right). \end{cases}$$

Il résulte immédiatement de ce rapprochement que chacune des formules de la trigonométrie rectiligne conduit à des formules analogues pour U_n et V_n, et inversement.

Ainsi la formule (3)

$$U_{2n} = U_n V_n,$$

correspond à la formule

$$\sin 2z = 2 \sin z \cos z;$$

les équations

$$(6)\qquad V_n + \delta U_n = 2a^n, \qquad V_n - \delta U_n = 2b^n,$$

que l'on déduit immédiatement des formules (2) correspondent exactement aux relations

$$\cos z + \sqrt{-1} \sin z = e^{z\sqrt{-1}}, \qquad \cos z - \sqrt{-1} \sin z = e^{-z\sqrt{-1}},$$

et les formules (4) sont entièrement analogues à celles qui ont été données dans les *Actes de Leipzick*, en 1701, par JEAN BERNOULLI, pour le développement de $\dfrac{\sin nz}{\sin z}$ et de $\cos nz$ suivant les puissances du sinus et du cosinus de l'arc z. Ainsi encore les formules

$$(7)\quad \begin{cases} [V_m + \delta U_m][V_n + \delta U_n] = 2[V_{m+n} + \delta U_{m+n}], \\ [V_n + \delta U_n]^r = 2^{r-1}[V_{nr} + \delta U_{nr}], \end{cases}$$

que l'on déduit des relations (6) coïncident avec les formules

$$(\cos x + \sqrt{-1} \sin x)(\cos y + \sqrt{-1} \sin y) = \cos(x+y) + \sqrt{-1} \sin(x+y),$$
$$(\cos x + \sqrt{-1} \sin x)^r = \cos rx + \sqrt{-1} \sin rx,$$

qui ont été données par MOIVRE.

Nous ferons encore observer que si, dans l'équation (1), on pose

$$X = x^r, \qquad \alpha = a^r, \qquad \beta = b^r,$$

les quantités α et β sont les racines de l'équation

$$(8)\qquad X^2 = V_r X - Q^r.$$

Par conséquent, chacune des formules qui appartiennent à la théorie présente peut être généralisée, en y remplaçant U_n et V_n par $\frac{U_{nr}}{U_r}$ et V_{nr}, P par V_r, Q par Q^r, et la différence δ des racines a et b par la différence δU_r des racines α et β de l'équation (8).

Les formules (4) deviennent ainsi

$$(9)\left\{\begin{aligned} 2^{n-1}\frac{U_{nr}}{U_r} &= \frac{n}{1}V_r^{n-1} + \frac{n(n-1)(n-2)}{1.2.3}\Delta U_r^2 V_r^{n-3} \\ &\quad + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}\Delta^2 U_r^4 V_r^{n-5} + \dots \\ 2^{n-1}V_{nr} &= V_r^n + \frac{n(n-1)}{1.2}\Delta U_r^2 V_r^{n-2} \\ &\quad + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\Delta^2 U_r^4 V_r^{n-4} + \dots \end{aligned}\right.$$

D'ailleurs, nous laisserons de côté, pour l'instant, les autres procédés de transformation de l'équation (1) par substitution de variable, ainsi que l'étude des fonctions plus générales

$$AU_n + BV_n + C,$$

dans lesquelles A, B, C désignent des nombres entiers quelconques, positifs ou négatifs.

SECTION III.

Des relations de récurrence pour le calcul des valeurs des fonctions U_n et V_n.

Le calcul des valeurs de U_n et de V_n qui correspondent aux valeurs entières et consécutives de n, s'effectue rapidement au moyen de formules entièrement analogues à celles de THOMAS SIMPSON :

$$\sin(n+2)z = 2\cos z \sin(n+1)z - \sin nz,$$
$$\cos(n+2)z = 2\cos z\cos(n+1)z - \cos nz.$$

En effet, multiplions par x^n les deux membres de l'équation (1), et remplaçons successivement x par a et b, nous obtenons

$$a^{n+2} = Pa^{n+1} - Qa^n, \qquad b^{n+2} = Pb^{n+1} - Qb^n,$$

et, par soustraction et par addition,

$$(10)\left\{\begin{aligned} U_{n+2} &= PU_{n+1} - QU_n, \\ V_{n+2} &= PV_{n+1} - QV_n. \end{aligned}\right.$$

Ces formules nous font voir que les fonctions U et V forment, pour les valeurs entières et consécutives de n, deux séries récurrentes de nombres entiers. Ces séries ont la même loi de formation, mais elles diffèrent par les conditions initiales. Nous généraliserons ces formules par l'emploi du calcul symbolique. En effet, en désignant par F une fonction quelconque, on tire évidemment de l'équation (1)

$$F(x^2) = F(Px - Q)\,;$$

si l'on remplace x par a et b, on a

$$a^n F(a^2) = a^n F(Pa - Q)\,, \qquad b^n F(b^2) = b^n F(Pb - Q)\,,$$

et, par soustraction et par addition, on obtient les égalités symboliques

$$(11)\quad \begin{cases} U^n F(U^2) = U^n F(PV - Q)\,, \\ V^n F(V^2) = V^n F(PV - Q)\,, \end{cases}$$

dans lesquelles on remplace, après le développement, les exposants de U et de V par des indices, en tenant compte de l'exposant zéro. Ainsi les symboles U^2 et $PU - Q$, V^2 et $PV - Q$ sont respectivement équivalents, et peuvent être remplacés l'un par l'autre dans les transformations algébriques.

On a, par exemple, dans la série de FIBONACCI, les résultats suivants

$$(12)\quad \begin{cases} u^{n+p} = u^{n-p}(u+1)^p\,, \\ u^{n-p} = u^n (u-1)^p\,, \end{cases}$$

qui sont entièrement analogues à ceux que l'on peut obtenir dans la théorie des combinaisons ou du triangle arithmétique, et, en particulier dans la formule du binôme des factorielles, due à VANDERMONDE.

En prenant, pour point de départ, l'équation

$$x^2 = x - 1\,,$$

on trouvera encore de nouvelles relations entre les coefficients de la même puissance du binôme.

La considération de l'équation (8) conduit aux relations suivantes

$$(13)\quad \begin{cases} U_{n+2r} = V_r U_{n+r} - Q^r U_n\,, \\ V_{n+2r} = V_r V_{n+r} - Q^r V_n\,, \end{cases}$$

qui permettent de calculer les valeurs des fonctions U_n et V_n qui correspondent à des valeurs de l'argument n en progression arithmétique de raison r.

Inversement, on trouvera, dans la théorie des fonctions circulaires et hyperboliques, des formules analogues aux formules (11) et (13).

SECTION IV.

Des relations des fonctions U_n et V_n avec les déterminants.

On peut exprimer U_n et U_{nr}, V_n et V_{nr} au moyen de déterminants; en effet, on a les formules

$$\begin{array}{rrrrrr}
 & & & & U_2 - PU_1 & = 0, \\
 & & & U_3 - PU_2 + & QU_1 & = 0, \\
 & & U_4 - PU_3 + & QU_2 & * & = 0, \\
 & U_5 - PU_4 + & QU_3 & * & * & = 0, \\
 \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
 \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
 U_{n+1} - PU_n + & QU_{n-1} & * & * & & = 0;
\end{array}$$

on en déduit

$$(14)\qquad U_{n+1} = (-1)^n \begin{vmatrix} -P, & +1, & 0, & 0, \ldots. \\ +Q, & -P, & +1, & 0, \ldots. \\ 0, & +Q, & -P, & +1, \ldots. \\ 0, & 0, & +Q, & -P, \ldots. \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} \;(n \text{ colonnes}),\ .$$

On obtient aussi

$$(15)\qquad V_n = (-1)^n \begin{vmatrix} -P, & +2, & 0, & 0, \ldots. \\ +Q, & -P, & +1, & 0, \ldots. \\ 0, & +Q, & -P, & +1, \ldots. \\ 0, & 0, & +Q, & -P, \ldots. \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} \;(n \text{ colonnes}),\ .$$

On vérifie les résultats que nous venons de trouver, en développant les déterminants suivant les éléments de la dernière ligne ou de la dernière colonne. Les valeurs de $\frac{U_{nr}}{U_r}$ et de V_{nr} s'obtiennent encore au moyen des déterminants, en remplaçant comme à l'ordinaire P par V_r, et Q par Q^r.

Enfin, nous ferons observer que ces formules sont susceptibles d'une grande généralisation; en effet, dans les formules (11) qui contiennent une fonction arbitraire, faisons n successivement égal à $1, 2, 3, \ldots. m$; nous obtenons alors m équations desquelles on tirera la valeur de l'une ou l'autre des fonctions U et V.

REMARQUE.— On peut encore pour le développement de U_n employer la formule suivante,

$$(16) \qquad U_{n+1} = \begin{vmatrix} P, & \sqrt{Q}, & 0, & 0, \ldots \\ \sqrt{Q}, & P, & \sqrt{Q}, & 0, \ldots \\ 0, & \sqrt{Q}, & P, & \sqrt{Q}, \ldots \\ 0, & 0, & \sqrt{Q}, & P, \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} \text{(n colonnes)};$$

cependant l'emploi de la formule (14) est bien préférable.

SECTION V.

Des relations des fonctions U_n et V_n avec les fractions continues.

Les fonctions U_n et V_n sont développables en fractions continues ; en effet, considérons l'expression

$$(17) \qquad \frac{R_n}{S_n} = a_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots + \cfrac{a_n}{b_n}}}},$$

et désignons par R_n et S_n le numérateur et le dénominateur de la $n^{\text{ième}}$ réduite ; on sait que l'on a

$$(18) \quad \begin{cases} R_{n+2} = b_{n-2} R_{n+1} + a_{n+2} R_n, \\ S_{n+2} = b_{n+2} S_{n+1} + a_{n+2} S_n ; \end{cases}$$

et, de plus

$$(19) \qquad R_n S_{n+1} - R_{n+1} S_n = (-1)^n a_1 a_2 a_3 \ldots a_{n+1}.$$

Par conséquent, si l'on pose

$$a_0 = b_1 = b_2 = \ldots = b_n = P,$$
$$a_1 = a_2 = a_3 = \ldots = a_n = -Q,$$

on obtient l'expression

$$(20) \qquad \frac{U_{n+1}}{U_n} = P - \cfrac{Q}{P - \cfrac{Q}{P - \cfrac{Q}{P - \cdots}}}$$

dans laquelle n désigne le nombre des quantités égales à P.

On a ainsi, dans la série de FIBONACCI :

$$\frac{1}{2}\,\frac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{(1+\sqrt{5})^{n}-(1-\sqrt{5})^{n}} = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}\;; \tag{21}$$

dans la série de FERMAT :

$$\frac{2^{n+1}-1}{2^{n}-1} = 3-\cfrac{2}{3-\cfrac{2}{3-\cfrac{2}{3-\ddots}}}\;; \tag{22}$$

et dans la série de PELL,

$$\frac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}{(1+\sqrt{2})^{n}-(1-\sqrt{2})^{n}} = 2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\ddots}}}}\,. \tag{23}$$

D'ailleurs, on a généralement

$$\frac{U_{n+1}}{U_n} = a\,\frac{1-\left(\frac{b}{a}\right)^{n+1}}{1-\left(\frac{b}{a}\right)^{n}}\;; \tag{24}$$

donc, en désignant par a la plus grande des racines, prises en valeur absolue, de l'équation (1), on a

$$\text{Lim}\,\frac{U_{n+1}}{U_n} = a\,, \tag{25}$$

lorsque n augmente indéfiniment. Cependant, nous ferons observer que ce dernier résultat ne s'applique pas dans le cas des séries de troisième espèce, c'est-à-dire lorsque les racines de l'équation proposée (1) sont imaginaires.

Au moyen de cette dernière formule, il est facile de calculer rapidement un terme de la série U_n lorsque l'on ne connait que le précédent. Soit, par exemple, dans la série de FIBONACCI

$$u_{44} = 7014\ 08733,$$

et

$$a = \frac{1+\sqrt{5}}{2} = 1,\ 61803\ 39887\ 39894\ 8482\ldots;$$

si l'on calcule par les méthodes abrégées le produit $a.u_{44}$, à moins d'une unité près, on trouve exactement, puisque u_n est entier

$$u_{45} = 11349\ 03170.$$

On peut, d'ailleurs, déterminer directement le dernier chiffre de u_n; ainsi dans ce cas particulier, il est facile de faire voir que deux termes, dont les rangs diffèrent d'un multiple quelconque de 60, sont terminés par le même chiffre; si l'on suppose alors p intérieur à 60, on peut démontrer que les derniers chiffres de u_p et de u_q sont complémentaires, lorsque la somme $p+q$ est égale à 60; on peut donc supposer maintenant p égal à 30; et même p inférieur à 15, si l'on observe que les termes u_{15+p} et u_{15-p} ont les mêmes derniers chiffres, lorsque p est impair, et leurs derniers chiffres complémentaires, lorsque p est pair.

On a, plus généralement, la formule

$$\frac{U_{(n+1)r}}{U_{nr}} = V_r - \cfrac{Q^r}{V_r - \cfrac{Q^r}{V_r - \cfrac{Q^r}{V_r - \ddots}}}, \tag{26}$$

dans laquelle les V_r sont en nombre n, et, lorsque n augmente indéfiniment,

$$\text{Lim}\ \frac{U_{(n+1)r}}{U_r} = a^r. \tag{27}$$

A la formule (26), correspond, dans la théorie des fonctions circulaires la formule

$$\frac{\sin\,(n+1)\,z}{\sin\,nz} = 2\cos z - \cfrac{1}{2\cos z - \cfrac{1}{2\cos z - \cfrac{1}{2\cos z - \ddots}}} \ * \tag{28}$$

dans laquelle l'expression $2\cos z$ est repétée n fois.

* Journal de Crelle, tome XVI, pag. 95; 1837.

On a aussi pour la série des V_n, la relation

$$(29) \qquad \frac{V_{nr}}{V_{n-1r}} = V_r - \cfrac{Q^r}{V_r - \cfrac{Q_r}{V_r - \cfrac{Q^r}{V_r - \cdots - \cfrac{Q^r}{\left(\frac{V_r}{2}\right)}}}},$$

dans laquelle la quantité V_r est répétée n fois.

Les nombreuses propriétés des déterminants et des fractions continues donnent lieu à des propriétés analogues pour les fonctions U_n et V_n. Ainsi la propriété bien connue de deux réduites consécutives, renfermée dans la formule (19) donne

$$(30) \quad \begin{cases} U_n^2 - U_{n-1}U_{n+1} = Q^{n-1}, \\ V_n^2 - V_{n-1}V_{n+1} = -Q^{n-1}\Delta, \end{cases}$$

et, plus généralement

$$(31) \quad \begin{cases} U_{nr}^2 - U_{(n-1)r}U_{(n+1)r} = Q^{(n-1)r}U_r^2, \\ V_{nr}^2 - V_{(n-1)r}V_{(n+1)r} = -Q^{(n-1)r}\Delta U_r^2; \end{cases}$$

on a, dans la théorie des fonctions circulaires, les formules analogues

$$\sin^2 x - \sin(x-y)\sin(x+y) = \sin^2 y,$$
$$\cos^2 x - \cos(x-y)\cos(x+y) = \sin^2 y,$$

Il est d'ailleurs facile de vérifier immédiatement les formules (31), en remplaçant U, V et Q en fonction de a et b. Ainsi, on a encore

$$\Delta U_{n+r}^2 = a^{2n+2r} + b^{2n+2r} - 2Q^{n+r},$$
$$\Delta U_n^2 \quad = a^{2n} \quad + b^{2n} \quad - 2Q^n \quad ;$$

donc, par soustraction :

$$\Delta\,[U_{n+r}^2 - Q^r U_n^2] = [a^{2n+r} - b^{2n+r}][a^r - b^r],$$

et, par suite

$$(32) \qquad U_{n+r}^2 - Q^r U_n^2 = U_r U_{2n+r};$$

on aura, par la même voie, la relation

$$(33) \qquad V_{n+r}^2 - Q^r V_n^2 = \Delta U_r U_{2n+r}.$$

La formule (32) donne plus particulièrement, pour $r = 1$, la relation

$$(34) \qquad U_{n+1}^2 - Q U_n^2 = U_{2n+1}.$$

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Cette dernière formule a été appliquée par M. GÜNTHER, à la résolution de l'équation indéterminée

$$y^2 - Qx^2 = Kz,$$

en nombres entiers *; il est facile de voir qu'un très-grand nombre de formules de cette section et des suivantes, conduisent à des conséquences analogues, mais beaucoup plus générales.

SECTION VI.

Développement des fonctions U_n et V_n en séries de fractions.

Les formules (30) donnent lieu aux développements de $\frac{U_{n+1}}{U_n}$ et $\frac{V_{n+1}}{V_n}$ en séries dont les termes ont pour dénominateurs le produit de deux termes consécutifs des séries U et V. On a, en effet,

$$\frac{U_{n+1}}{U_n} = \frac{U_2}{U_1} + \left(\frac{U_3}{U_2} - \frac{U_2}{U_1}\right) + \left(\frac{U_4}{U_3} - \frac{U_3}{U_2}\right) \ldots + + \left(\frac{U_{n+1}}{U_n} - \frac{U_n}{U_{n-1}}\right),$$

et, en réunissant les fractions contenues dans chaque parenthèse,

$$(35) \qquad \frac{U_{n+1}}{U_n} = \frac{U_2}{U_1} - \frac{Q}{U_1 U_2} - \frac{Q^2}{U_2 U_3} - \frac{Q^3}{U_3 U_4} - \ldots - \frac{Q_{n-1}}{U_{n-1} U_n};$$

on a ainsi dans la série de FIBONACCI, pour n augmentant indéfiniment

$$(36) \qquad \frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1.1} - \frac{1}{1.2} + \frac{1}{2.3} - \frac{1}{3.5} + \frac{1}{5.8} - \frac{1}{8.13} + \ldots.$$

En suivant la même voie, on obtient les formules plus générales

$$(37) \qquad \frac{U_{(n+1)r}}{U_{nr}} = \frac{U_{2r}}{U_r} - \left[\frac{Q^r}{U_r U_{2r}} + \frac{Q^{2r}}{U_{2r} U_{3r}} + \ldots + \frac{Q^{(n-1)r}}{U_{(n-1)r} U_{nr}}\right] U_r^2,$$

et

$$(38) \qquad \frac{V_{(n+1)r}}{V_{nr}} = \frac{V_r}{V_0} - \left[\frac{Q^r}{V_0 V_r} + \frac{Q^{2r}}{V_r V_{2r}} + \ldots + \frac{Q^{nr}}{V_{(n-1)r} V_{nr}}\right] \Delta U_r^2.$$

On tire encore des deux relations

$$(39) \qquad \begin{cases} U_{n+r} V_n - U_n V_{n+r} = 2Q^n U_r, \\ V_{n+r} V_n - \Delta U_n U_{n+r} = 2Q^n V_r, \end{cases}$$

* *Journal de Mathématiques pures et appliquées,* de M. RESAL, pag. 331–341; Octobre, 1876.

que nous démontrerons plus loin, les développements

$$(40)\quad \begin{cases} \dfrac{U_{n+kr}}{V_{n+kr}} = \dfrac{U_n}{V_n} + 2Q^n U_r \left[\dfrac{1}{V_r V_{n+r}} + \dfrac{Q^r}{V_{n+r} V_{n+2r}} + \cdots + \dfrac{Q^{(k-1)r}}{V_{n+(k-1)r} V_{n+kr}} \right], \\ \dfrac{V_{n+kr}}{U_{n+kr}} = \dfrac{V_n}{U_n} - 2Q^n U_r \left[\dfrac{1}{U_r U_{n+r}} + \dfrac{Q^r}{U_{n+r} U_{n+2r}} + \cdots + \dfrac{Q^{(k-1)r}}{U_{n+(k-1)r} U_{n+kr}} \right]. \end{cases}$$

Lorsque k augmente indéfiniment, les premiers membres des égalités précédentes ont respectivement pour limites $\dfrac{1}{\sqrt{\Delta}}$ et $\sqrt{\Delta}$; on tiendra compte dans le second membre, des conditions de convergence.

On peut ainsi développer la racine carrée d'un nombre entier en séries de fractions ayant pour numérateur l'unité; c'était un usage familier aux savants de la Grèce et de l'Egypte; ainsi, par exemple, cette valeur approximative de

$$\frac{\sqrt{3}}{4} = \frac{1}{3} + \frac{1}{10} + \varepsilon,$$

rapportée par COLUMELLE au chapitre V de son ouvrage *de Rê Rusticâ;* ainsi encore, cette valeur approximative de

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{12.34} + \varepsilon,$$

donnée par les auteurs indiens BAUDHAYANA et APASTAMBA*; cette valeur approximative est égale au rapport des termes $V_8 = 577$ et $U_8 = 408$, de la série de PELL.

SECTION VII.

Des relations des fonctions U_n et V_n avec la théorie de la divisibilité.

Si nous posons $\alpha = a^r$ et $\beta = b^r$, et, par suite, $\alpha\beta = Q^r$, nous obtenons, par la formule qui donne le quotient de $\alpha^n - \beta^n$ par $\alpha - \beta$, les résultats suivants:

1°. Lorsque n désigne un nombre *pair :*

$$(41)\quad \frac{U_{nr}}{U_r} = V_{(n-1)r} + Q^r V_{(n-3)r} + Q^{2r} V_{(n-5)r} + \cdots + Q^{\left(\frac{n}{2}-1\right)r} V_r;$$

2°. Lorsque n désigne un nombre *impair :*

$$(42)\quad \frac{U_{nr}}{U_r} = V_{(n-1)r} + Q^r V_{(n-3)r} + Q^{2r} V_{(n-5)r} + \cdots + Q^{\frac{n-1}{2}r}.$$

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* *The Çulvasûtras* by G. THIBAUT, pag. 13-15. *Journal of the Asiatic Society of Bengal*, 1875.

Le quotient de $\alpha^n - \beta^n$ par $\alpha + \beta$, lorsque n désigne un nombre *pair*, donne encore

$$(43) \qquad \frac{U_{nr}}{V_r} = U_{(n-1)r} - Q^r U_{(n-3)r} + Q^{2r} U_{(n-5)r} - \ldots + (-Q^r)^{\frac{n}{2}-1} U_r,$$

et le quotient de $\alpha^n + \beta^n$ par $\alpha + \beta$, lorsque n désigne un nombre *impair*, donne enfin

$$(44) \qquad \frac{V_{nr}}{V_r} = V_{(n-1)r} - Q^r V_{(n-3)r} + Q^{2r} V_{(n-5)r} - \ldots + (-Q^r)^{\frac{n-1}{2}}.$$

Pour $n = 2$, on retrouve la formule

$$(3) \qquad U_{2r} = U_r V_r,$$

et, pour $n = 3$, on a

$$(45) \qquad \begin{cases} U_{3r} = U_r (V_{2r} + Q^r), \\ V_{3r} = V_r (V_{2r} - Q^r). \end{cases}$$

Les relations précédentes nous montrent que U_m est toujours divisible par U_n, lorsque m est divisible par n; de même V_m est toujours divisible par V_n, lorsque m est impair et divisible par n; par conséquent U_m et V_m ne peuvent être des nombres premiers, que si m est premier; mais la réciproque de ce théorème n'a pas lieu.

Dans la série de FIBONACCI, u_3 est divisible par 2, u_4 est divisible par 3, u_5 est divisible par 5; par conséquent, u_{3n}, u_{4n} et u_{5n} sont respectivement divisibles par 2, 3, et 5. Ainsi encore, bien que 53 soit premier on a

$$u_{53} = 953 \times 559\,45741.$$

Reprenons les égalités

$$(6) \qquad V_n + \delta U_n = 2a^n, \quad V_n - \delta U_n = 2b^n;$$

nous obtenons, en multipliant membre à membre, la relation

$$(46) \qquad V_n^2 - \Delta U_n^2 = 4Q^n,$$

qui correspond, en trigonométrie, à la formule

$$\cos^2 z + \sin^2 z = 1.$$

Cette relation nous montre que si U_n et V_n admettaient un diviseur commun θ, ce diviseur serait un facteur de Q; mais, d'autre part,

$$V_n = \left(\frac{P+\delta}{2}\right)^n + \left(\frac{P-\delta}{2}\right)^n,$$

et, en supprimant les multiples de Q, ce qui revient évidemment à remplacer δ par Q, on a la congruence.

$$(47) \qquad V_n \equiv P^n, \qquad (\text{Mod. } Q);$$

donc, tout diviseur θ de U_n et V_n diviserait P et Q; or nous avons supposé premiers entre eux. De là résulte cette proposition :

THÉORÈME : *Les nombres U_n et V_n sont premiers entre eux.*

Si l'on désigne par μ l'exposant auquel appartient P suivant le module Q, on sait que la congruence

$$P^n \equiv 1, \qquad (\text{Mod. } Q),$$

est vérifiée pour toutes les valeurs de n égales à un multiple quelconque de μ, μ étant lui-même un certain diviseur de l'indicateur $\varphi(Q)$ de Q, ou du nombre des entiers inférieurs et premiers à Q; par conséquent, à cause de l'égalité (47), on résoudra la congruence

$$V_n \equiv 1, \qquad (\text{Mod. } Q), \tag{48}$$

par toutes les valeurs de n égales à un multiple quelconque de μ.

SECTION VIII.

Des formes linéaires et quadratiques des diviseurs de U_n et V_n, qui correspondent aux valeurs paires et impaires de l'argument n.

La formule (46) conduit encore à d'autres conséquences importantes sur la forme des diviseurs de U_n et de V_n, car on en déduit immédiatement les propositions suivantes, suivant que l'on considère n égal à un nombre *pair*, ou à un nombre *impair*.

THÉORÈME : *Les termes de rang impair de la série U_n sont des diviseurs de la forme quadratique $x^2 - Qy^2$.*

En tenant compte des résultats bien connus de la théorie des diviseurs des formes quadratiques, on a, en particulier, pour les formes linéaires correspondantes des diviseurs premiers impairs de U_{2r+1}

dans la série de	FIBONACCI :	$4q + 1$;
" "	FERMAT :	$8q + 1, 7$;
" "	PELL :	$4q + 1$.

Ainsi, les termes de rang impair de la série de FIBONACCI ou de la série de PELL ne peuvent contenir comme diviseur aucun nombre premier de la forme $4q + 3$.

THÉORÈME : *Les termes de rang pair de la série V_n sont des diviseurs de la forme quadratique $x^2 + \Delta y^2$.*

En particulier, les formes linéaires correspondantes des diviseurs premiers impairs de V_{2r} sont

dans la série de FIBONACCI : $20q + 1, 3, 7, 9$;
" " FERMAT : $4q + 1$;
" " PELL : $8q + 1, 3$.

THÉORÈME : *Les termes de rang impair de la série V_n sont des diviseurs de la forme quadratique $x^2 + Q\Delta y^2$.*

En particulier, les formes linéaires correspondantes des diviseurs premiers impairs de V_{2r+1} sont

dans la série de FIBONACCI : $20q + 1, 9, 11, 19$;
" " FERMAT : $8q + 1, 3$;
" " PELL : $8q + 1, 7$.

SECTION IX.

Des formules concernant l'addition des fonctions numériques.

En multipliant membre à membre les relations

$$V_m + \delta U_m = 2a^m, \qquad V_n + \delta U_n = 2a^n,$$

on obtient,

$$V_m V_n + \Delta U_m U_n + \delta [U_m V_n + U_n V_m] = 4a^{m+n};$$

si l'on change a en b, et δ en $-\delta$, on déduit ensuite par addition et par soustraction, les formules

$$(49)_r \begin{cases} 2U_{m+n} = U_m V_n + U_n V_m, \\ 2V_{m+n} = V_m V_n + \Delta U_m U_n, \end{cases}$$

auxquelles correspondent en trigonométrie les formules de l'addition des arcs :

$$\sin(x+y) = \sin x \cos y + \sin y \cos x,$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

Si nous changeons n en $-n$ dans les formules (49), en tenant compte des relations

$$(50) \qquad U_{-n} = -\frac{U_n}{Q^n}, \qquad V_{-n} = \frac{V_n}{Q^n},$$

nous obtenons

$$(51) \begin{cases} 2Q^n U_{m-n} = U_m V_n - U_n V_m, \\ 2Q^n V_{m-n} = V_m V_n - \Delta U_m U_n; \end{cases}$$

en faisant $m = n + r$, on obtient les formules (39) données plus haut.

La comparaison des égalités (49) et (51) nous donne immédiatement

$$U_{m+n} + Q^n U_{m-n} = U_m V_n,$$
$$U_{m+n} - Q^n U_{m-n} = U_n V_m;$$

posons maintenant

$$m + n = r,\quad m - n = S,$$

il vient

$$(52)\quad \begin{cases} U_r + Q^{\frac{r-s}{2}} U_s = U_{\frac{r+s}{2}} V_{\frac{r-s}{2}}, \\ U_r - Q^{\frac{r-s}{2}} U_s = U_{\frac{r-s}{2}} V_{\frac{r+s}{2}}; \end{cases}$$

ces relations sont entièrement semblables à celles qui permettent de transformer la somme ou la différence de deux lignes trigonométriques, en un produit. On a, de même

$$(53)\quad \begin{cases} V_r + Q^{\frac{r-s}{2}} V_s = V_{\frac{r+s}{2}} V_{\frac{r-s}{2}}, \\ V_r - Q^{\frac{r-s}{2}} V_s = \Delta U_{\frac{r+s}{2}} U_{\frac{r-s}{2}}. \end{cases}$$

On aura encore, comme pour la somme des sinus ou des cosinus d'arcs en progression arithmétique

$$(54)\quad \begin{cases} U_m + Q^{\frac{-r}{2}} U_{m+r} + Q^{\frac{-2r}{2}} U_{m+2r} + \ldots + Q^{\frac{-nr}{2}} U_{m+nr} = U_{\frac{2m+nr}{2}} \dfrac{U_{\frac{n+1}{2}r} Q^{\frac{m}{4}}}{U_{\frac{r}{2}} Q^{\frac{nr}{2}}}, \\ V_m + Q^{\frac{-r}{2}} V_{m+r} + Q^{\frac{-2r}{2}} V_{m+2r} + \ldots + Q^{\frac{-nr}{2}} V_{m+nr} = V_{\frac{2m+nr}{2}} \dfrac{U_{\frac{n+1}{2}r} Q^{\frac{m}{4}}}{U_{\frac{r}{2}} Q^{\frac{nr}{2}}}; \end{cases}$$

et, par suite

$$(55)\quad \frac{U_m + Q^{\frac{-r}{2}} U_{m+r} + Q^{\frac{-2r}{2}} U_{m+2r} + \ldots + Q^{\frac{-nr}{2}} U_{m+nr}}{V_m + Q^{\frac{-r}{2}} V_{m+r} + Q^{\frac{-2r}{2}} V_{m+2r} + \ldots + Q^{\frac{-nr}{2}} V_{m+nr}} = \frac{U_{m+\frac{n}{2}r}}{V_{m+\frac{n}{2}r}}.$$

On trouve des formules beaucoup plus simples en partant des relations

$$(13)\quad \begin{cases} U_{n+2r} = V_r U_{n+r} - Q^r U_n, \\ V_r V_{n+2r} = V_r V_{n+r} - Q^r V_n; \end{cases}$$

si l'on remplace successivement n par 0, r, $2r$, ... $(n-1)\,r$, et si l'on ajoute, on obtient

$$(56)\quad \begin{cases} U_r + U_{2r} + \ldots\ldots + U_{nr} = \dfrac{U_r + Q^n U_{nr} - U_{(n+1)r}}{1 + Q^r - V_r}, \\ V_r + V_{2r} + \ldots\ldots + V_{nr} = \dfrac{V_r + Q^n V_{nr} - V_{(n+1)r}}{1 + Q^r - V_r}. \end{cases}$$

Ces formules se présentent sous une forme indéterminée lorsque le dénominateur s'annule, c'est-à-dire pour

$$1 + a^r b^r - a^r - b^r = 0,$$

ou bien

$$(1 - a^r)(1 - b^r) = 0;$$

c'est-à-dire pour les valeurs de a ou de b égales à l'unité; dans ce cas, on emploie le procédé de sommation de la progression géometrique. On a d'ailleurs, dans la série de FIBONACCI, pour $r = 1$ et $r = 2$,

$$\begin{aligned} u_1 + u_2 + u_3 + \ldots + u_n &= u_{n+2} - 1, \\ u_2 + u_4 + u_6 + \ldots + u_{2n} &= u_{2n+1} - 1, \\ v_1 + v_2 + v_3 + \ldots + v_n &= v_{n+2} - 3, \\ v_2 + v_4 + v_6 + \ldots + v_{2n} &= v_{2n+1} - 1. \end{aligned}$$

On trouvera encore plus généralement,

$$(57) \quad U_{m+r} + U_{m+2r} + U_{m+3r} + \ldots + U_{m+nr} = \frac{U_{m+r} + Q^r U_{m+nr} - U_{m+(n+1)r} - Q^r U_m}{1 + Q^r - V_r},$$

et un résultat analogue en changeant U en V.

La formule d'addition peut s'écrire encore

$$2\frac{U_{m+n}}{U_n} = \frac{U_m}{U_n} V_n + V_m;$$

on a, par conséquent

$$(58) \quad 2\frac{U_{m+n} U_{m+n-1} \ldots U_{m+1}}{U_n \; U_{n-1} \ldots U_1} = \frac{U_{m+n-1} U_{m+n-2} \ldots U_m}{U_n \; U_{n-1} \ldots U_1} V_n + \frac{U_{m+n-1} U_{m+n-2} \ldots U_{m+1}}{U_{n-1} \; U_{n-2} \ldots U_1} V_m.$$

On en déduit immédiatement cette proposition:

THÉORÈME: *Le produit de n termes consécutifs de la série* U_n *est divisible par le produit des n premiers termes.*

Nous terminerons ce paragraphe par la démonstration de formules d'une extrême importance; car elles nous serviront ultérieurement comme base de la théorie des fonctions numériques doublement périodiques, déduites de la considération des fonctions symétriques des racines des équations du troisième et du quatrième degré à coefficients commensurables. Les formules (30) nous donnent

$$\begin{aligned} U_{m-1} U_{m+1} &= U_m^2 - {}^{m-1}Q, \\ U_{n-1} U_{n+1} &= U_n^2 - {}^{n-1}Q; \end{aligned}$$

on en déduit

$$U_n^2 U_{m-1} U_{m+1} - U_m^2 U_{n-1} U_{n+1} = Q^{n-1}[U_m^2 - Q^{m-n} U_n^2],$$

et, par les formules (32)

$$(A) \qquad U_n^2 U_{m-1} U_{m+1} - U_m^2 U_{n-1} U_{n+1} = Q^{n-1} U_{m-n} U_{m+n};$$

on a, de même

$$(A') \qquad V_n^2 V_{m-1} V_{m+1} - V_m^2 V_{n-1} V_{n+1} = -\Delta Q^{n-1} V_{m-n} V_{m+n}.$$

En particulier, pour $m = n + 1$, et pour $m = n + 2$, on a

$$(B) \quad \begin{cases} U_n^3 U_{n+2} - U_{n+1}^3 U_{n-1} = Q^{n-1} U_1 U_{2n+1}, \\ U_n^2 U_{n+1} U_{n+3} - U_{n+2}^2 U_{n-1} U_{n+1} = Q^{n-1} U_2 U_{2n+2}, \end{cases}$$

et des formules analogues pour les V_n.

Les formules (A) et (B) appartiennent à la théorie des fonctions elliptiques, et, plus spécialement, aux fonctions que Jacobi a désignées par les symboles Θ et H.

Section X.

De la somme des carrés des fonctions numériques U_n et V_n.

Si dans la relation suivante

$$(59) \qquad \Delta U_{r+2\kappa\rho} U_{s+2\kappa\sigma} = V_{r+s+2\kappa(\rho+\sigma)} - Q^{s+2\kappa\sigma} V_{r-s+2\kappa(\rho-\sigma)},$$

nous supposons successivement κ égal à $0, 1, 2, 3, \ldots n$, et si nous ajoutons membre à membre les égalités obtenues, après avoir divisé respectivement par

$$1, \quad Q^{\rho+\sigma}, \quad Q^{2(\rho+\sigma)}, \ldots . Q^{n(\rho+\sigma)}, \ldots$$

nous obtenons la formule

$$(60) \quad \begin{cases} \displaystyle\sum_{\kappa=0}^{\kappa=n} U_{r+2\kappa\rho} U_{s+2\kappa\sigma} = \frac{U_{r+2n\rho} U_{s+(2n+1)\sigma} - Q^{\sigma-\rho} U_{r+(2n+1)\rho} U_{s+2n\sigma}}{\Delta Q^{n(\rho+\sigma)} U_{\sigma-\rho} U_{\sigma+\rho}} \\ \qquad\qquad + \dfrac{U_r U_{s-2\sigma} - Q^{\sigma-\rho} U_{r-2\rho} U_s}{\Delta U_{\sigma-\rho} U_{\sigma+\rho}}. \end{cases}$$

On a, en particulier, pour $2\rho = r$ et $2\sigma = s$,

$$(61) \quad \begin{cases} U_r U_s + \dfrac{U_{2r} U_{2s}}{Q^{\frac{r+s}{2}}} + \dfrac{U_{3r} U_{3s}}{Q^{r+s}} + \ldots + \dfrac{U_{(n+1)r} U_{(n+1)s}}{Q^{\frac{n}{2}(r+s)}} \\ \qquad = \dfrac{U_{(n+1)r} U_{2(n+1)s} - Q^{\frac{s-r}{2}} U_{(n+1)s} U_{2(n+1)r}}{\Delta Q^{\frac{n}{2}(r+s)} U_{\frac{s-r}{2}} U_{\frac{s+r}{2}}}, \end{cases}$$

et, plus particuliérement encore

$$(62)\quad \begin{cases} \dfrac{U_r^2}{Q^r}+\dfrac{U_{2r}^2}{Q^{2r}}+\dfrac{U_{3r}^2}{Q^{3r}}+\ldots+\dfrac{U_{(n+1)r}^2}{Q^{(n+1)r}}=\dfrac{1}{\Delta}\left[\dfrac{U_{(2n+3)r}}{U_rQ^{(n+1)r}}-2n-3\right], \\ \dfrac{U_r^2}{Q^r}+\dfrac{U_{3r}^2}{Q^{3r}}+\dfrac{U_{5r}^2}{Q^{5r}}+\ldots+\dfrac{U_{(2n+1)r}^2}{Q^{(2n+1)r}}=\dfrac{1}{\Delta}\left[\dfrac{U_{4(n+1)r}}{U_{2r}Q^{(2n+1)r}}-2n-2\right]. \end{cases}$$

Par un procédé analogue, on trouvera aussi les valeurs de

$$\sum_{\kappa=0}^{\kappa=n}\frac{V_{r+2\kappa\rho}V_{s+2\kappa\sigma}}{Q^{\kappa(\rho+\sigma)}}\quad\text{et de}\quad\sum_{\kappa=0}^{\kappa=n}\frac{U_{r+2\kappa\rho}V_{s+2\kappa\sigma}}{Q^{\kappa(\rho+\sigma)}}.$$

En particulier

$$(63)\quad \begin{cases} \dfrac{V_r^2}{Q^r}+\dfrac{V_{2r}^2}{Q^{2r}}+\dfrac{V_{3r}^2}{Q^{3r}}+\ldots+\dfrac{V_{nr}^2}{Q^{nr}}=2n-1+\dfrac{U_{(2n+1)r}}{U^rQ^{nr}}, \\ \dfrac{V_r^2}{Q^r}+\dfrac{V_{3r}^2}{Q^{3r}}+\dfrac{V_{5r}^2}{Q^{5r}}+\ldots+\dfrac{V_{(2n+1)r}^2}{Q^{(2n+1)r}}=2n+2+\dfrac{U_{4(n+1)r}}{U_rQ^{(2n+1)r}}. \end{cases}$$

On a aussi, dans le cas général,

$$(64)\quad \begin{cases} \displaystyle\sum_{\kappa=0}^{\kappa=n}U_{m+\kappa r}^2=\frac{V_{2m+2(n+1)r}-V_{2m}-Q^{2r}[V_{2m+2nr}-V_{2m-2r}]}{\Delta(V_{2r}-Q^{2r}-1)}-2Q^m\frac{Q^{(n+1)r}-1}{\Delta(Q^r-1)} \\ \displaystyle\sum_{\kappa=0}^{\kappa=n}V_{m+\kappa r}^2=\frac{V_{2m+2(n+1)r}-V_{2m}-Q^{2r}[V_{2m+2nr}-V_{2m-2r}]}{V_{2r}-Q_{2r}-1}+2Q^m\frac{Q^{(n+1)r}-1}{Q^r-1} \end{cases}$$

On a, par exemple, dans la série de FIBONACCI

$$(65)\quad \begin{cases} u_r^2-u_{2r}^2+u_{3r}^2-\ldots-(-1)^{nr}u_{nr}^2=\dfrac{1}{5}\left[2n+1-(-1)^{nr}\dfrac{u_{(2n+1)r}}{u_r}\right], \\ u_r^2+u_{3r}^2+u_{5r}^2+\ldots+u_{(2n+1)r}^2=\dfrac{1}{5}\left[\dfrac{u_{4(n+1)r}}{u_{2r}}-(-1)^{nr}(2n+2)\right], \\ v_r^2-v_{2r}^2+v_{3r}^2-\ldots-(-1)^nv_{nr}^2=2n-1-(-1)^{nr}\dfrac{u_{(2n+1)r}}{u_r}, \\ v_r^2+v_{3r}^2+v_{5r}^2+\ldots+v_{(n+1)r}^2=\dfrac{u_{4(n+1)r}}{u_{2r}}-(-1)^r(2n-2); \end{cases}$$

la formule plus simple

$$(66)\qquad u_1^2+u_2^2+u_3^2+\ldots+u_n^2=u_nu_{n+1},$$

donne ainsi, pour le côté du décagone régulier étoilé, cette expression

$$(67)\quad \begin{cases} \dfrac{1+\sqrt{5}}{2}=1+\dfrac{1}{1^2}-\dfrac{1}{1^2+1^2}+\dfrac{1}{1^2+1^2+2^2}-\dfrac{1}{1^2+1^2+2^2+3^2} \\ \qquad\qquad+\dfrac{1}{1^2+1^2+2^2+3^2+5^2}-\ldots. \end{cases}$$

On a encore, dans cette série

$$(68)\quad \begin{cases} u_1u_2+u_2u_3+u_3u_4+\ldots\ldots+u_{2n-1}u_{2n}=u_{2n}^2\,, \\ u_1u_2+u_2u_3+u_3u_4+\ldots\ldots+u_{2n}u_{2n+1}=u_{2n+1}^2-1. \end{cases}$$

SECTION XI.

Des relations des fonctions U_n et V_n avec la théorie du plus grand commun diviseur.

Nous avons trouvé la formule

$$2U_{m+n}=U_mV_n+U_nV_m\,;$$

par conséquent, si un nombre impair quelconque θ divise U_{m+n} et U_m, il divise U_nV_m; mais nous avons démontré (§ 17) que U_m et V_m sont premiers entre eux; donc θ divise U_n. Inversement, tout nombre impair qui divise U_n et U_m divise U_{m+n}; donc, en ne tenant pas compte du facteur 2, on a cette proposition fondamentale:

THÉORÈME: *Le plus grand commun diviseur de U_m et de U_n est égal à U_D, en désignant par D le plus grand commun diviseur de m et de n.*

En particulier, les termes U_m et U_n sont premiers entre eux lorsque m et n sont premiers entre eux, car U_1 est égal à l'unité. On déduit d'ailleurs du théorème fondamental un grand nombre de propositions entièrement semblables à celles que l'on obtient dans la théorie du plus grand commun diviseur et du plus petit multiple commun de plusieurs nombres donnés.

Il résulte encore de ce qui précède que, dans la recherche du plus grand commun diviseur de deux termes U_m et U_n, les restes successifs forment aussi des termes de la série; en particulier, les restes successifs de deux termes consécutifs donnent, dans le cas de Q négatif, tous les termes de la série décroissante, à partir du plus petit d'entre eux. LAMÉ * a observé que, dans la recherche du plus grand commun diviseur de deux nombres quelconques, le nombre des restes est au plus égal au nombre des termes de la série de FIBONACCI, inférieurs au plus petit des deux nombres donnés, et il en a déduit ce théorème:

Le nombre des divisions à effectuer dans la recherche du plus grand commun diviseur de deux nombres donnés est au plus égal, dans le système ordinaire de numération, à cinq fois le nombre des chiffres du plus petit des deux nombres donnés.

* Comptes rendus de l'Académie des Sciences de Paris, t. xix, pag. 868. Paris, 1844.

On trouverait une limite plus rapprochée, en calculant par logarithmes le rang du terme de la série de FIBONACCI immediatement inférieur au plus petit des nombres donnés. On voit aisément qu'il suffirait, en désignant ce plus petit nombre par A, de prendre le plus petit entier contenu dans la fraction

$$\frac{\log A + \log \sqrt{5}}{\log \frac{1+\sqrt{5}}{2}} = \frac{\log A + 0.349}{0.209}.$$

Mais il est préférable de s'en tenir à la limite donnée par l'élégant théorème que nous venons de rappeler.

SECTION XII.

De la multiplication des fonctions numériques.

On peut exprimer les valeurs de U_n et V_n qui correspondent à toutes les valeurs entières et positives de n, en fonction des valeurs initiales; en effet, on a successivement, pour U, par exemple,

$$(69)\quad \left\{\begin{aligned} &U_2 = PU_1 - QU_0, \\ &U_3 = (P^2 - Q)\,U_1 - QPU_0, \\ &U_4 = (P^3 - 2PQ)\,U_1 - Q\,(P^2 - Q)\,U_0, \\ &U_5 = (P^4 - 3P^2Q + Q^2)\,U_1 - Q\,(P^3 - 2PQ)\,U_0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right.$$

On observera d'abord que si ϕ_n désigne le coefficient de U_1 dans U_{n+1}, on a en général,

$$U_{n+1} = \phi_n U_1 - Q\phi_{n-1} U_0.$$

Le coefficient ϕ_n est une fonction homogène et de degré n, de P et de Q, en y considérant P au premier degré et Q au second. Si l'on forme le tableau des coefficients de ϕ_n, on retrouve aisément le triangle arithmétique, mais dans une disposition spéciale. On a d'ailleurs, ainsi qu'on peut le vérifier *a posteriori*

$$(70)\quad \left\{\begin{aligned} \phi_n = P^n - \frac{n-1}{1} P^{n-2}Q + \frac{(n-2)(n-3)}{1.2} P^{n-4}Q^2 & \\ - \frac{(n-3)(n-4)(n-5)}{1.2.3} P^{n-6}Q^3 + \ldots, & \end{aligned}\right.$$

et, en même temps

$$(71)\quad \begin{cases} U_{n+1} = \phi_n U_1 - Q\phi_{n-1} U_0, \\ V_{n+1} = \phi_n V_1 - Q\phi_{n-1} V_0, \end{cases}$$

avec les conditions initiales

$$U_0 = 0, \quad U_1 = 1, \quad V_0 = 2, \quad V_1 = P;$$

par conséquent, on a encore

$$\phi_n = U_{n+1}.$$

On a, en particulier, dans la série de FIBONACCI, pour $P = 1$ et $Q = -1$,

$$(72)\quad 1 + C_{n-1,1} + C_{n-2,2} + C_{n-3,3} + \ldots\ldots = u_n,$$

et, pour $P = 1$, $Q = 1$,

$$(73)\quad 1 - C_{n-1,1} + C_{n-2,2} - C_{n-3,3} + \ldots = \frac{2}{\sqrt{3}} \sin \frac{n\pi}{3}.$$

Les formules précédentes se généralisent aisément par la considération de l'équation (8). En effet, si l'on pose

$$(74)\quad \psi_n = V_r^n - \frac{n-1}{1} V_r^{n-2} Q^r + \frac{(n-2)(n-3)}{1.2} V_r^{n-4} Q^{2r} - \frac{(n-3)(n-4)(n-5)}{1.2.3} V_r^{n-6} Q^{3r} + \ldots.$$

on obtient, comme ci-dessus,

$$(75)\quad \begin{cases} U_{m+2nr} = \psi_{n-1} U_{m+r} - Q^r \psi_{n-2} U_m, \\ V_{m+2nr} = \psi_{n-1} V_{m+r} - Q^r \psi_{n-2} V_m, \end{cases}$$

et, pour $m = 0$, on a encore la relation

$$(76)\quad \psi_{n-1} = \frac{U_{2nr}}{U_r}$$

qui permet de calculer inversement la fonction ψ à l'aide des valeurs de U. D'ailleurs, cette relation, dans laquelle n désigne un nombre entier, a lieu quelle que soit la valeur de r; on a ainsi, pour $r = 0$, la formule

$$(77)\quad n = 2^{n-1} - C_{n-2,1} 2^{n-3} + C_{n-3,2} 2^{n-5} - C_{n-4,3} 2^{n-7} + \ldots.$$

Nous ferons observer que les résultats précédents correspondent aux développements bien connus de $\frac{\sin nz}{\sin z}$ et de $\cos nz$ suivant les puissances de $\cos z$, obtenus pour la première fois par VIÈTE.*

* OPERA, Leyde, 1646, pag. 295-299.

SECTION XIII.

De la loi de la répétition des nombres premiers dans les séries récurrentes simplement périodiques.

Nous exprimerons encore les fonctions U_{np} et V_{np}, en fonctions entières de U_n et de V_n, par des formules analogues à celles qui ont été données par MOIVRE et par LAGRANGE.*

En effet, si l'on désigne par C_m^n le nombre des combinaisons de m objets pris n à n, on a la relation suivante:

$$(78)\quad \alpha^p + \beta^p = (\alpha+\beta)^p - \frac{p}{1}\alpha\beta(\alpha+\beta)^{p-2} + \frac{p}{2}C_{p-3}^1\alpha^2\beta^2(\alpha+\beta)^{p-4} + \dots$$
$$+ (-1)^r\frac{p}{r}C_{p-r-1}^{r-1}\alpha^r\beta^r(\alpha+\beta)^{p-2r} + \dots,$$

que l'on peut vérifier *à posteriori*, et dans laquelle tous les coefficients sont entiers, puisque l'on a

$$\frac{p}{r}C_{p-r-1}^{r-1} = C_{p-r-1}^r + C_{p-r-1}^{r-1}.$$

Posons, dans l'hypothèse de p *impair*,

$$\alpha = a^n \text{ et } \beta = -b^n,$$

nous obtenons

$$(79)\quad U_{np} = \delta^{p-1}U_n^p + \frac{p}{1}Q^n\delta^{p-3}U_n^{p-2} + \frac{p}{2}C_{p-3}^1Q^{2n}\delta^{p-5}U_n^{p-4} + \dots$$
$$+ \frac{p}{r}C_{p-r-1}^{r-1}Q^{nr}\delta^{p-2r-1}U_n^{p-2r} + \dots.$$

La formule précédente conduit à la loi de la répétition des nombres premiers dans les séries récurrentes que nous considérons ici. Dans la série naturelle des nombres entiers, un nombre premier p apparaît pour la première fois, à son rang, et à la première puissance; il arrive à la seconde puissance au rang p^2, à la troisième au rang p^3, et ainsi de suite; de plus, tous les termes divisibles par p^a occupent un rang égal à un multiple quelconque de p^a. Mais dans les séries récurrentes simplement périodiques, il n'en est pas complètement ainsi. Nous démontrons plus loin que les termes de celles-ci contiennent, à des rangs déterminés, tous les nombres premiers;

* Commentarii Acad. Petrop., t. XIII, ad annum MDCCXLI-XLIII, pag. 29. Leçons sur le calcul des fonctions, pag. 119.

mais si ces nombres premiers p n'apparaissent pas, pour la première fois, dans la série au rang p, cependant ils s'y reproduisent à intervalles égaux à p, comme dans la série ordinaire, et l'apparition de leurs puissances successives se fait comme dans la série naturelle. Ainsi, en général, dans l'étude arithmétique des séries, deux lois sont à considérer: la *loi de l'apparition* des nombres premiers, et la *loi de la répétition.*

Nous démontrerons, pour l'instant, que la loi de la répétition est identiquement la même dans la série naturelle, et dans les séries des U_n. En effet, si p désigne un nombre premier, et U_n le premier terme de la série divisible par p^λ, on observera que le dernier terme de la formule précédente est divisible par $p^{\lambda+1}$, et non par une puissance supérieure de p; on a donc la proposition fondamentale suivante:

THÉORÈME: *Si λ désigne le plus grand exposant d'un nombre premier p contenu dans U_n, l'exposant de la plus haute puissance de p, qui divise U_{pn}, est égal à* $\lambda + 1$.

Ainsi, par exemple, dans la série de FIBONACCI, u_8 est divisible par 7; donc u_{56} est divisible par 7^2 et non par 7^3; dans la série de PELL, u_7 et u_{30} sont respectivement divisibles par 13^2 et par 31^2; donc U_{91} et U_{930} sont divisibles par 13^3 et par 31^3, et non par des puissances supérieures.

Inversement, si $a^p \pm b^p$ est divisible par p^λ, $a \pm b$ est divisible par $p^{\lambda-1}$; ce résultat donne des conséquences importantes dans la théorie de l'équation indéterminée

$$x^p + y^p + z^p = 0,$$

dont l'irrésolubilité, non démontrée jusqu'à présent, constitue la dernière proposition de FERMAT.

SECTION XIV.

Nouvelles formes linéaires et quadratiques des diviseurs de U_n et de V_n.

La formule (79) donne, successivement, pour p égal à 3, 5, 7, 9, ... les formules suivantes

$$(80)\quad \left\{\begin{aligned}
U_{3n} &= \Delta\, U_n^3 + 3Q^n U_n,\\
U_{5n} &= \Delta^2 U_n^5 + 5Q^n\Delta\, U_n^3 + 5Q^{2n} U_n,\\
U_{7n} &= \Delta^3 U_n^7 + 7Q^n\Delta^2 U_n^5 + 14Q^{2n}\Delta\, U_n^3 + 7Q^{3n} U_n,\\
U_{9n} &= \Delta^4 U_n^9 + 9Q^n\Delta^3 U_n^7 + 27Q^{2n}\Delta^2 U_n^5 + 30Q^{3n}\Delta\, U_n^3 + 9Q^{4n} U_n,\\
U_{11n} &= \Delta^5 U_n^{11} + 11Q^n\Delta^4 U_n^9 + 44Q^{2n}\Delta^3 U_n^7 + 77Q^{3n}\Delta^2 U_n^5 + 55Q^{4n}\Delta\, U_n^3\\
&\quad + 11Q^{5n} U_n,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right.$$

On a ainsi

(81) $$\frac{U_{3n}}{U_n} = \Delta U_n^2 + 3Q^n,$$

et, par suite, la proposition suivante:

THÉORÈME: *Les diviseurs de* $\frac{U_{3n}}{U_n}$ *sont des diviseurs de la forme quadratique* $\Delta x^2 + 3Q^n y^2$.

En particulier, les formes linéaires des diviseurs premiers impairs de $\frac{U_{6n}}{U_{2n}}$ sont

pour la série de FIBONACCI: $30q + 1, 17, 19, 23$;
de FERMAT: $6q + 1$;
de PELL: $24q + 1, 5, 7, 11$;

et les formes linéaires des diviseurs premiers impairs de $\frac{U_{3(2n+1)}}{U_{2n+1}}$ sont

pour la série de FIBONACCI: $60q + 1, 7, 11, 17, 43, 49, 53, 59$;
de FERMAT: $24q + 1, 5, 7, 11$;
de PELL: $24q + 1, 5, 19, 23$.

On a aussi

(82) $$4\frac{U_{5n}}{U_n} = (2\Delta U_n^2 + 5Q^n)^2 - 5Q^{2n},$$

et, par suite:

THÉORÈME: *Les diviseurs de* $\frac{U_{5n}}{U_n}$ *sont des diviseurs de la forme quadratique* $x^2 - 5y^2$.

Les formes linéaires des diviseurs premiers impairs sont, dans les trois séries prises pour exemples,

$$20q + 1, 9, 11, 19.$$

Nous avons aussi

(83) $$4\frac{U_{7n}}{U_n} = \Delta\,[2\Delta U_n^3 + 7Q^n U_n]^2 + 7Q^{2n} V_n^2,$$

et, par suite:

THÉORÈME: *Les diviseurs de* $\frac{U_{7n}}{U_n}$ *sont des diviseurs de la forme quadratique* $\Delta x^2 + 7y^2$.

Supposons maintenant que p désigne un nombre *pair*, et faisons encore, dans la formule (78),

$$\alpha = a^n,\ \beta = -b^n,$$

nous obtenons

$$(84)\qquad V_{np} = \delta^p U_n^p + \frac{p}{1} Q^n \delta^{p-2} U_n^{p-2} + \frac{p}{2} C_{p-3}^1 Q^{2n} \delta^{p-4} U^{p-4} + \dots$$
$$+ \frac{p}{r} C_{p-r-1}^{r-1} Q^{nr} \delta^{p-2r} U_n^{p-2r} + \dots$$

On a, en particulier, pour $p = 2$, la formule

$$(85)\qquad V_{2n} = \Delta U_n^2 + 2Q^n,$$

et, par conséquent, la proposition suivante:

THÉORÈME: *Les diviseurs de* V_{2n} *sont des diviseurs de la forme quadratique* $\Delta x^2 + 2Q^n y^2$.

Les formes linéaires correspondantes des diviseurs premiers impairs sont, pour *n pair*

dans la série de FIBONACCI: $40q + 1, 7, 9, 11, 13, 19, 23, 37$;
de FERMAT: $8q + 1, 3$;
de PELL: $4q + 1$;

et, pour *n impair*

dans la série de FIBONACCI: $40q + 1, 3, 9, 13, 27, 31, 37, 39$;
de FERMAT: $4q + 1$;
de PELL: $4q + 1$.

On devra, dans les applications, combiner ces résultats avec ceux que nous avons donnés dans la Section VIII.

Faisons enfin dans la formule (79),

$$\alpha = a^n, \qquad \beta = b^n,$$

nous obtenons, en supposant indifféremment que p est égal à un nombre *pair* ou à un nombre *impair:*

$$(86)\qquad V_{np} = V_n^p - \frac{p}{1} Q^n V_n^{p-2} + \frac{p}{2} C_{p-3}^1 Q^{2n} V_n^{p-4} - \dots$$
$$+ (-1)^r \frac{p}{r} C_{p-r-1}^{r-1} Q^{nr} V_n^{p-2r} + \dots$$

On a ainsi, en faisant successivement p égal à 2, 3, 4, 5, 6, ... les résultats suivants

$$(87)\qquad \left\{ \begin{aligned} V_{2n} &= V_n^2 - 2Q^n, \\ V_{3n} &= V_n^3 - 3Q^n V_n, \\ V_{4n} &= V_n^4 - 4Q^n V_n^2 + 2Q^{2n}, \\ V_{5n} &= V_n^5 - 5Q^n V_n^3 + 5Q^{2n} V_n, \\ V_{6n} &= V_n^6 - 6Q^n V_n^4 + 9Q^{2n} V_n^2 - 2Q^{3n}, \\ & \dots\dots\dots\dots \end{aligned} \right.$$

qui conduisent encore à des formules entièrement semblables aux précédentes.

SECTION XV.

Des relations des fonctions U_n et V_n avec les radicaux continus.

On tire de l'équation

$$x^2 = Px - Q,$$

la formule

$$x = \sqrt{-Q + Px},$$

et, successivement

$$x = \sqrt{-Q + P\sqrt{-Q + Px}}$$

$$x = \sqrt{-Q + P\sqrt{-Q + P\sqrt{-Q + Px}}}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ;$$

par conséquent, puisque l'on peut supposer P positif, on a, pour Q négatif,

$$(88) \qquad a = \text{Lim.} \sqrt{-Q + P\sqrt{-Q + P\sqrt{-Q + \cdots}}},$$

a désignant la racine positive de l'équation proposée. Ainsi, dans la série de FIBONACCI

$$\frac{1+\sqrt{5}}{2} = \text{Lim.} \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}},$$

dans la série de PELL,

$$1 + \sqrt{2} = \text{Lim.} \sqrt{1 + 2\sqrt{1 + 2\sqrt{1 + 2\sqrt{1 + \cdots}}}},$$

et, dans la série de FERMAT,

$$2 = \text{Lim.} \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}.$$

On sait que ce dernier radical se présente dans le calcul de π, par la méthode des périmètres, imaginée par ARCHIMÈDE.

Mais les résultats obtenus dans la section précédente, conduisent à des formules plus importantes, qui trouveront leur emploi dans la recherche des grands nombres premiers. On tire, par exemple, de la première des formules (87)

$$V_n = \sqrt{2Q^n + V_{2n}};$$

et, de même, en changeant n en $2n$, $4n$, $8n$, ...

$$V_{2n} = \sqrt{2Q^{2n} + V_{4n}},$$

$$V_{4n} = \sqrt{2Q^{4n} + V_{8n}},$$

$$V_{8n} = \sqrt{2Q^{8n} + V_{16n}};$$

et, par suite,

$$(89)\quad \begin{cases} V_n = \sqrt{2Q^n + V_{2n}}, \\ V_n = \sqrt{2Q^n + \sqrt{2Q^{2n} + V_{4n}}}, \\ V_n = \sqrt{2Q^n + \sqrt{2Q^{2n} + \sqrt{2Q^{4n} + V_{8n}}}}, \\ V_n = \sqrt{2Q^n + \sqrt{2Q^{2n} + \sqrt{2Q^{4n} + \sqrt{2Q^{8n} + V_{16n}}}}}, \\ \dots\dots\dots\dots\dots, \end{cases}$$

et ainsi indéfiniment. Ces formules sont analogues à celles que l'on obtient pour $\cos\frac{\pi}{4}$, $\cos\frac{\pi}{8}$, $\cos\frac{\pi}{16}$, $\cos\frac{\pi}{32}$, .. $\cos\frac{\pi}{2^r}$.

La seconde des relations (87) donne de la même façon

$$(90)\quad \begin{cases} V_n = \sqrt{3Q^n + \dfrac{V_{3n}}{V_n}}, \\ V_n = \sqrt{2Q^n + \sqrt{3Q^{2n} + \dfrac{V_{6n}}{V_{2n}}}}, \\ V_n = \sqrt{2Q^n + \sqrt{2Q^{2n} + \sqrt{3Q^{4n} + \dfrac{V_{12n}}{V_{4n}}}}}, \end{cases}$$

ces formules sont semblables à celles que l'on obtient pour $\cos\frac{\pi}{6}$, $\cos\frac{\pi}{12}$, $\cos\frac{\pi}{24}$, ... $\cos\frac{\pi}{3.2^r}$.

La troisième des relations (87) conduit encore à des formules qui correspondent à celles qui donnent $\cos\frac{\pi}{10}$, $\cos\frac{\pi}{20}$, $\cos\frac{\pi}{40}$, .. $\cos\frac{\pi}{5.2^r}$; et ainsi de quelques autres.

SECTION XVI.

Développements des puissances de U_n et de V_n en fonctions linéaires des termes dont les arguments sont des multiples de n.

On peut exprimer les puissances de U_n et de V_n en fonctions linéaires des termes dont les rangs sont des multiples de n, par des formules analogues à celles qui donnent les puissances de sin z et de cos z, développées suivant les sinus et les cosinus des multiples de l'arc z. En désignant d'abord par p un nombre *impair*, le développement de $(\alpha - \beta)^p$, donne

$$(\alpha-\beta)^p = (\alpha^p - \beta^p) - \frac{p}{1}\alpha\beta(\alpha^{p-2} - \beta^{p-2}) + \frac{p(p-1)}{1.2}\alpha^2\beta^2(\alpha^{p-4} - \beta^{p-4}) - \dots$$

et, par suite, en faisant

$$\alpha = a^n, \qquad \beta = b^n,$$

on obtient la formule

$$(91)\quad \Delta^{\frac{p-1}{2}} U_n^p = U_{pn} - \frac{p}{1} Q^n U_{(p-2)n} + \frac{p(p-1)}{1.2} Q^{2n} U_{(p-4)n}$$

$$-\frac{p(p-1)(p-2)}{1.2.3} Q^{3n} U_{(p-6)n} + \ldots + \ldots \pm \frac{p(p-1)\ldots\frac{p+3}{2}}{1.2\ldots\frac{p-1}{2}} Q^{\frac{p-1}{2}n} U_n.$$

On a successivement, pour p égal à 3, 5, 7, 9, ...

$$(92)\quad \begin{cases} \Delta\, U_n^3 = U_{3n} - 3Q^n U_n, \\ \Delta^2 U_n^5 = U_{5n} - 5Q^n U_{3n} + 10Q^{2n} U_n, \\ \Delta^3 U_n^7 = U_{7n} - 7Q^n U_{5n} + 21Q^{2n} U_{3n} - 35Q^{4n} U_n, \\ \Delta^4 U_n^9 = U_{9n} - 9Q^n U_{7n} + 36Q^{2n} U_{5n} - 84Q^{4n} U_{3n} + 126Q^{6n} U_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Le développement de $(\alpha - \beta)^p$ donne encore, en supposant maintenant que p désigne un nombre *pair*:

$$(93)\quad \Delta^{\frac{p}{2}} U_n^p = V_{pn} - \frac{p}{1} Q^n V_{(p-2)n} + \frac{p(p-1)}{1.2} Q^{2n} V_{(p-4)n}$$

$$-\frac{p(p-1)(p-2)}{1.2.3} Q^{3n} V_{(p-6)n} + \ldots + \ldots \pm \frac{p(p-1)\ldots\left(\frac{p}{2}+1\right)}{1.2\ldots\frac{p}{2}} Q^{\frac{p}{2}n},$$

et, pour p successivement égal à 2, 4, 6, 8, ...

$$(94)\quad \begin{cases} \Delta\, U_n^2 = V_{2n} - 2Q^n, \\ \Delta^2 U_n^4 = V_{4n} - 4Q^n V_{2n} + 6Q^{2n}, \\ \Delta^3 U_n^6 = V_{6n} - 6Q^n V_{4n} + 15Q^{2n} V_{2n} - 20Q^{3n}, \\ \Delta^4 U_n^8 = V_{8n} - 8Q^n V_{6n} + 28Q^{2n} V_{4n} - 56Q^{3n} V_{2n} + 70Q^{4n}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Le développement de $(\alpha + \beta)^p$ donne, dans l'hypothèse de p égal à un nombre *impair*,

$$(95)\quad V_n^p = V_{pn} + \frac{p}{1} Q^n V_{(p-2)n} + \frac{p(p-1)}{1.2} Q^{2n} V_{(p-4)n}$$

$$+\frac{p(p-1)(p-2)}{1.2.3} Q^{3n} V_{(p-6)n} + \ldots + \ldots + \frac{p(p-1)\ldots\frac{p+3}{2}}{1.2\ldots\frac{p-1}{2}} Q^{\frac{p-1}{2}n} V_n,$$

et, plus particulièrement

$$(96)\quad \begin{cases} V_n^3 = V_{3n} + 3Q^n V_n, \\ V_n^5 = V_{5n} + 5Q^n V_{3n} + 10Q^{2n} V_n, \\ V_n^7 = V_{7n} + 7Q^n V_{5n} + 21Q^{2n} V_{3n} + 35Q^{3n} V_n, \\ V_n^9 = V_{9n} + 9Q^n V_{7n} + 36Q^{2n} V_{5n} + 84Q^{3n} V_{3n} + 126Q^{4n} V_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

De même, lorsque p désigne un nombre *pair*,

$$(97)\quad V_n^p = V_{pn} + \frac{p}{1} Q^n V_{(p-2)n} + \frac{p(p-1)}{1.2} Q^{2n} V_{(p-4)n}$$

$$+ \frac{p(p-1)(p-2)}{1.2.3} Q^{3n} V_{(p-6)n} + \ldots + \ldots + \frac{p(p-1)\ldots\left(\frac{p}{2}+1\right)}{1.2\ldots.\left(\frac{p}{2}\right)} Q^{\frac{p}{2}n};$$

on a, plus particulièrement,

$$(98)\quad \begin{cases} V_n^2 = V_{2n} + 2Q^n, \\ V_n^4 = V_{4n} + 4Q^n V_{2n} + 6Q^{2n}, \\ V_n^6 = V_{6n} + 6Q^n V_{4n} + 15Q^{2n} V_{2n} + 20Q^{3n}, \\ V_n^8 = V_{8n} + 8Q^n V_{6n} + 28Q^{2n} V_{4n} + 56Q^{3n} V_{2n} + 70Q^{4n}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Les relations (91), (93), (95) et (97) sont elles mêmes des cas particuliers des formules suivantes:

$$(99)\quad \begin{cases} V_r^n U_{(m-n)r} = U_{mr} + \frac{n}{1} Q^r U_{(m-2)r} + \frac{n(n-1)}{1.2} Q^{2r} U_{(m-4)r} \\ \qquad\qquad + \frac{n(n-1)(n-2)}{1.2.3} Q^{3r} U_{(m-6)r} + \ldots. \\ V_r^n V_{(m-n)r} = V_{mr} + \frac{n}{1} Q^r V_{(m-2)r} + \frac{n(n-1)}{1.2} Q^{2r} V_{(m-4)r} \\ \qquad\qquad + \frac{n(n-1)(n-2)}{1.2.3} Q^{3r} V_{(m-6)r} + \ldots. \\ \Delta^n U_r^{2n} U_{(m-2n)r} = U_{mr} - \frac{2n}{1} Q^r U_{(m-2)r} + \frac{2n(2n-1)}{1.2} Q^{2r} U_{(m-4)r} - \ldots. \\ \Delta^n U_r^{2n} V_{(m-2n)r} = V_{mr} - \frac{2n}{1} Q^r V_{(m-2)r} + \frac{2n(2n-1)}{1.2} Q^{2r} V_{(m-4)r} - \ldots. \\ \Delta^n U_r^{2n+1} U_{(m-2n-1)r} = U_{mr} - \frac{2n+1}{1} Q^r U_{(m-2r)} + \frac{(2n+1).2n}{1.2} Q^{2r} U_{(m-4)r} - \ldots \\ \Delta^n U_r^{2n+1} V_{(m-2n-1)r} = V_{mr} - \frac{2n+1}{1} Q^r V_{(m-2r)} + \frac{(2n+1).2n}{1.2} Q^{2r} V_{(m-4)r} - \ldots \end{cases}$$

Ces relations trouvent principalement leur emploi dans la sommation des puissances semblables des fonctions U_n et V_n. Le développement de la puissance d'un binôme donne encore lieu à un certain nombre d'autres. Ainsi, on a, par exemple

$$\alpha = \overline{\alpha + \beta} - \beta \quad \text{et} \quad \beta = \overline{\beta + \alpha} - \alpha\,;$$

donc, pour p égal à un nombre *impair*

$$\alpha^p + \beta^p = (\alpha+\beta)^p - \frac{p}{1}\beta(\alpha+\beta)^{p-1} + \frac{p(p-1)}{1.2}\beta^2(\alpha+\beta)^{p-2} + \ldots + \frac{p}{1}\beta^{p-1}(\alpha+\beta),$$

$$\alpha^p + \beta^p = (\alpha+\beta)^p - \frac{p}{1}\alpha(\alpha+\beta)^{p-1} + \frac{p(p-1)}{1.2}\alpha^2(\alpha+\beta)^{p-2} + \ldots + \frac{p}{1}\alpha^{p-1}(\alpha+\beta)\,;$$

on a ainsi, en ajoutant et en retranchant, après avoir posé $\alpha = a^n$, $\beta = b^n$, les formules suivantes :

$$(100)\quad \begin{cases} 2V_{np} = V_0 V_n^p - \dfrac{p}{1} V_n V_n^{p-1} + \dfrac{p(p-1)}{1.2} V_{2n} V_n^{p-2} + \ldots + \dfrac{p}{2} V_{(p-1)n} V_n, \\ 0 = \dfrac{p}{1} U_n V_n^{p-1} - \dfrac{p(p-1)}{1.2} U_{2n} V_n^{p-2} + \ldots - \dfrac{p}{1} U_{(p-1)n} V_n. \end{cases}$$

On trouvera des développements analogues pour p égal à un nombre pair, et d'autres encore à l'aide des identités

$$\alpha = \overline{\alpha - \beta} + \beta \text{ et } \beta = \overline{\beta - \alpha} + \alpha.$$

La formule suivante, que l'on peut déduire du *Problème des partis*

$$(\alpha+\beta)^{p+q-1}$$
$$= \alpha^p\left[(\alpha+\beta)^{q-1} + \frac{p}{1}(\alpha+\beta)^{q-2}\beta + \frac{p(p+1)}{1.2}(\alpha+\beta)^{q-3}\beta^2 + \ldots + C_{p+q-2}^{q-1}\beta^{q-1}\right]$$
$$+ \beta^q\left[(\alpha+\beta)^{p-1} + \frac{q}{1}(\alpha+\beta)^{p-2}\alpha + \frac{q(q+1)}{1.2}(\alpha+\beta)^{q-3}\alpha^2 + \ldots + C_{p+q-2}^{p-1}\alpha^{q-1}\right]$$

donne en changeant α en β, puis par addition et par soustraction,

$$(101)\quad \begin{cases} 2V_n^{p+q-1} = V_{pn}V_n^{q-1} + \dfrac{p}{1} Q^n V_{(p-1)n} V_n^{q-2} + \dfrac{p(p+1)}{1.2} Q^{2n} V_{(p-2)n} V_n^{q-3} \\ \qquad + \ldots + C_{p+q-2}^{q-1} Q^{(q-1)n} V_{(p-q+1)n} + V_{qn} V_n^{p-1} + \dfrac{q}{1} Q^n V_{(q-1)n} V_n^{p-2} \\ \qquad + \dfrac{q(q+1)}{1.2} Q^{2n} V_{(q-2)n} V_n^{p-3} + \ldots + C_{p+q-2}^{p-1} Q^{(p-1)} V^{(q-p+1)n}, \\ 0 = U_{pn} V_n^{q-1} + \dfrac{p}{1} Q^n U_{(p-1)n} V_n^{q-2} + \dfrac{p(p+1)}{1.2} Q^{2n} U_{(p-2)n} V_n^{q-3} \\ \qquad + \ldots + C_{p+q-2}^{q-1} Q^{(q-1)n} U_{(p-q+1)n} - U_{qn} V_n^{p-1} - \dfrac{q}{1} Q^n U_{(q-1)n} V_n^{p-2} \\ \qquad - \dfrac{q(q+1)}{1.2} Q^{2n} U_{(q-2)n} V_n^{p-3} - \ldots - C_{p+q-2}^{p-1} Q^{(p-1)n} U_{(q-p+1)n}. \end{cases}$$

On obtiendrait deux autres formules semblables aux précédentes, en posant $\alpha = a^n$ et $\beta = b^n$; on simplifie ces formules, en faisant $p = q$.

SECTION XVII.

Autres formules concernant le développement des fonctions numériques U_n et V_n.

Considérons les fonctions α et β de z,

$$\alpha = \left(\frac{z + \sqrt{z^2 - 4h}}{2}\right)^n, \qquad \beta = \left(\frac{z - \sqrt{z^2 - 4h}}{2}\right)^n;$$

on tire, en différentiant

$$\frac{d\alpha}{\alpha dz} = \frac{n}{\sqrt{z^2 - 4h}},$$

et, en faisant disparaître le radical

$$(z^2 - 4h)\frac{d\alpha^2}{dz^2} - n^2\alpha^2 = 0.$$

Une nouvelle différentiation nous donne

$$(z^2 - 4h)\frac{d^2\alpha}{dz^2} + z\frac{d\alpha}{dz} - n^2\alpha = 0;$$

il est d'ailleurs facile de voir que les fonctions β, $\alpha + \beta$ et $\alpha - \beta$ vérifient la même équation différentielle. On a donc, en désignant par $f(z)$ l'une quelconque d'entre elles, par l'application du théorème de LEIBNIZ

$$(z^2 - 4h)\frac{d^{p+2}f(z)}{dz^{p+2}} + (2p + 1)z\frac{d^{p+1}f(z)}{dz^{p+1}} + (p^2 - n^2)\frac{d^pf(z)}{dz^p} = 0,$$

et, pour $z = 0$,

$$4h\frac{d^{p+2}f(0)}{dz^{p+2}} = (p^2 - n^2)\frac{d^pf(0)}{dz^p}.$$

Si l'on suppose $z = V_r$, $h = Q^r$, la formule de MACLAURIN nous donne, pour *n pair*, les deux développements

$$(102)\quad \begin{cases} \dfrac{V_{nr}}{2(-Q^r)^{\frac{n}{2}}} = 1 - \dfrac{n^2}{1.2}\dfrac{V_r^2}{2^2Q^r} + \dfrac{n^2(n^2-2^2)}{1.2.3.4}\dfrac{V_r^4}{2^4Q^{2r}} - \dfrac{n^2(n^2-2^2)(n^2-4^2)}{1.2.3.4.5.6}\dfrac{V_r^6}{2^6Q^{3r}} + \dots, \\[2ex] \dfrac{-U_{nr}}{2(-Q^r)^{\frac{n}{2}}U_r} = \dfrac{n}{1}\dfrac{V_r}{2Q^r} - \dfrac{n(n^2-2^2)}{1.2.3}\dfrac{V_r^3}{2^3Q^{3r}} + \dfrac{n(n^2-2^2)(n^2-4^2)}{1.2.3.4.5}\dfrac{V_r^5}{2^5Q^{5r}} - \dots \end{cases}$$

et, pour *n impair*

$$(103)\quad \begin{cases} \dfrac{U_{nr}}{(-Q^r)^{\frac{n-1}{2}}} = U_r\left[1 - \dfrac{n^2-1^2}{1.2}\dfrac{V_r^2}{2^2Q^r} + \dfrac{(n^2-1^2)(n^2-3^2)}{1.2.3.4}\dfrac{V_r^4}{2^4Q^{2r}} - \dots\right], \\[2ex] \dfrac{V_{nr}}{(-Q^r)^{\frac{n-1}{2}}} = V_r\left[n - \dfrac{n(n^2-1^2)}{1.2.3}\dfrac{V_r^2}{2^2Q^r} + \dfrac{n(n^2-1^2)(n^2-3^2)}{1.2.3.4.5}\dfrac{V_r^4}{2^4Q^{2r}} - \dots\right]. \end{cases}$$

On peut d'ailleurs vérifier ces formules, et les suivantes, *à posteriori*, en observant que si l'on pose

$$G_{m,\varkappa} = (m^2 - 2^2)(m^2 - 4^2) \dots (m^2 - 4\varkappa^2)\,,$$
$$H_{m,\varkappa} = (m^2 - 1^2)(m^2 - 3^2) \dots \big(m^2 - (2\varkappa - 1)^2\big)\,,$$

on a les relations

$$mG_{m,\varkappa} = (m - 2\varkappa)\, H_{m+1,\varkappa} = (m + 2\varkappa)\, H_{m-1,\varkappa}\,.$$

Au lieu de développer les fonctions U_{nr} et V_{nr} suivant les puissances de V_r, on peut aussi les développer suivant les puissances de U_r; on trouve ainsi, pour n *pair*

$$(104)\quad \left\{\begin{aligned} \frac{V_{nr}}{2Q^{\frac{nr}{2}}} &= 1 + \frac{n^2}{1.2}\frac{\Delta U_r^2}{2^2Q^r} + \frac{n^2(n^2-2^2)}{1.2.3.4}\frac{\Delta^2 U_r^4}{2^4Q^{2r}} + \frac{n^2(n^2-2^2)(n^2-4^2)}{1.2.3.4.5}\frac{\Delta^3 U_r^6}{2^6Q^{3r}} + \dots, \\ \frac{U_{nr}}{Q^{(\frac{n}{2}-1)r}} &= \frac{U_{2r}}{2}\left[n + \frac{n(n^2-2^2)}{1.2.3}\frac{\Delta U_r^2}{2^2Q^r} + \frac{n(n^2-2^2)(n^2-4^2)}{1.2.3.4.5}\frac{\Delta^2 U_r^4}{2^4Q^{2r}} + \dots\right], \end{aligned}\right.$$

et, pour n *impair*

$$(105)\quad \left\{\begin{aligned} \frac{V_{nr}}{Q^{\frac{n-1}{2}r}} &= V_r\left[1 + \frac{n^2-1^2}{1.2}\frac{\Delta U_r^2}{2^2Q^r} + \frac{(n^2-1^2)(n^2-3^2)}{1.2.3.4}\frac{\Delta^2 U_r^4}{2^4Q^{2r}}\right. \\ &\qquad\left. + \frac{(n^2-1^2)(n^2-3^2)(n^2-5^2)}{1.2.3.4.5.6}\frac{\Delta^3 U_r^6}{2^6Q^{3r}} + \dots\right], \\ \frac{U_{nr}}{Q^{\frac{n-1}{2}r}} &= U_r\left[n + \frac{n(n^2-1^2)}{1.2.3}\frac{\Delta U_r^2}{2^2Q^r} + \frac{n(n^2-1^2)(n^2-3^2)}{1.2.3.4.5}\frac{\Delta^2 U_r^4}{2^4Q^{2r}}\right. \\ &\qquad\left. + \frac{n(n^2-1^2)(n^2-3^2)(n^2-5^2)}{1.2.3.4.5.6.7}\frac{\Delta^3 U_r^6}{2^6Q^{3r}} + \dots\right]. \end{aligned}\right.$$

En ayant égard à l'une ou l'autre des relations

$$V_{nr}^2 = V_{2nr} + 2Q^{nr}\,, \quad \text{et} \quad \Delta U_{nr}^2 = V_{2nr} - 2Q^{nr}\,,$$

on obtiendra de nouvelles formules, et ainsi, par exemple:

$$(106)\quad \frac{U_{nr}^2}{Q^{(n-2)r}U_r^2} = n^2 - \frac{n^2(n^2-1^2)}{3.4}\frac{\Delta U_r^2}{Q^r} + \frac{n^2(n^2-1^2)(n^2-2^2)}{3.4.5.6}\frac{\Delta^2 U_r^4}{{}^{2r}Q} - \dots$$

On peut d'ailleurs mettre cette dernière formule et quelques autres sous une forme assez remarquable, en observant que l'on a, pour m quelconque et n entier positif, l'identité

$$\frac{m^2(m^2-1^2)(m^2-3^2)\dots(m^2-(n-1)^2)}{3.4.5\,\dots\dots\,2n} = \frac{(m-n)(m-n+1)(m-n+2)\dots(m+n-1)}{1.2.3\,\dots\dots\,(2n)}$$
$$+ \frac{(m-n+1)(m-n+2)\dots(m+n)}{1.2.3\,\dots\dots\,(2n)}\,.$$

Par conséquent, les coefficients de la formule (106) sont entiers, et l'on a

$$\frac{n^2(n^2-1^2)(n^2-2^2)\ldots(n^2-\overline{r-1}^2)}{3.4.5\ldots\ldots(2r)} = C^{2r}_{n+r-1} + C^{2r}_{n+r}.\ *$$

Nous ferons observer que les formules (104) et (105) subsistent encore pour des valeurs quelconques de n; on a alors des développements en séries convergentes, lorsque $\frac{\Delta U_r^2}{2^2 Q^r}$ n'est pas supérieur à l'unité; en effet, si l'on pose

$$\frac{\Delta U_r^2}{2^2 Q^r} \leqslant 1,$$

le rapport d'un terme au précédent finit par devenir négatif (pour Δ positif), et inférieur à l'unité en valeur absolue. Cette condition est remplie pour $r=1$ dans la série de PELL; on a donc, quelle que soit la valeur de n

$$(107)\quad \begin{cases} \frac{1}{2}\left[(\sqrt{2}+1)^n+(\sqrt{2}-1)^n\right] = 1 + \frac{n^2}{1.2} + \frac{n^2(n^2-2^2)}{1.2.3.4} \\ \qquad\qquad + \frac{n^2(n^2-2^2)(n^2-4^2)}{1.2.3.4.5.6} + \ldots, \\ \frac{1}{2}\left[(\sqrt{2}+1)^n+(\sqrt{2}-1)^n\right] = n + \frac{n(n^2-1^2)}{1.2.3} + \frac{n(n^2-1^2)(n^2-3^2)}{1.2.3.4.5} + \ldots. \end{cases}$$

SECTION XVIII.

Développements en séries des irrationnelles et de leurs logarithmes népériens.

Les développements des fonctions en séries, par la formule de MACLAURIN, donnent lieu à un très-grand nombre de formules nouvelles, pour le développement des fonctions numériques que nous considérons ici, et par suite, pour celui des fonctions circulaires et hyperboliques. Lorsque les séries correspondantes ne sont convergentes que pour les valeurs de la variable dont le module est inférieur à une limite donnée, on peut toujours supposer que cette variable x est choisie de telle sorte que la série représente la fonction, pour toutes les valeurs de x dont le module est inférieur à l'unité. Soit donc la série

$$F(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + \ldots;$$

* En désignant par a le résidu de n^2 suivant le module p, premier avec n, on déduit de cette identité une démonstration immédiate d'une proposition contenue au No. 128 des *Disquisitiones Arithmeticæ*.

on aura, en supposant z positif,

$$F\left(\frac{z}{1+z}\right) = A_0 + A_1\frac{z}{1+z} + A_2\frac{z^2}{(1+z)^2} + A_3\frac{z^3}{(1+z)^3} + \ldots,$$

$$F\left(\frac{1}{1+z}\right) = A_0 + A_1\frac{1}{1+z} + A_2\frac{1}{(1+z)^2} + A_3\frac{1}{(1+z)^3} + \ldots$$

et, par conséquent:

$$F\left(\frac{1}{1+z}\right) + F\left(\frac{z}{1+z}\right) = 2A_0 + A_1\frac{1+z}{1+z} + A_2\frac{1+z^2}{(1+z)^2} + A_3\frac{1+z^3}{(1+z)^3} + \ldots,$$

$$F\left(\frac{1}{1+z}\right) - F\left(\frac{z}{1+z}\right) = A_1\frac{1-z}{1+z} + A_2\frac{1-z^2}{(1+z)^2} + A_3\frac{1-z^3}{(1+z)^3} + \ldots.$$

Si l'on désigne par a la plus grande des racines, supposée positive, de l'équation fondamentale (1), par r un nombre *pair*, ou un nombre entier quelconque, suivant que la racine b est négative ou positive, et si l'on pose $z = \frac{b^r}{a^r}$, on obtient

$$(108)\quad \begin{cases} F\left(\dfrac{a^r}{a^r+b^r}\right) + F\left(\dfrac{a^rb^r}{a^r+b^r}\right) = 2A_0 + A_1\dfrac{V_r}{V_r} + A_2\dfrac{V_{2r}}{V_r^2} + A_3\dfrac{V_{3r}}{V_r^3} + \ldots, \\ F\left(\dfrac{a^r}{a^r+b^r}\right) - F\left(\dfrac{a^rb^r}{a^r+b^r}\right) = \sqrt{\Delta}\left[A_1\dfrac{U_r}{V_r} + A_2\dfrac{U_{2r}}{V_r^2} + A_3\dfrac{U_{3r}}{V_r^3} + \ldots\right]. \end{cases}$$

Si l'on suppose $z = -\frac{b^r}{a^r}$, on obtient deux développements analogues aux précédents; ces développements sont parfois, très-lentement convergents; mais leur étude conduit à des propriétés importantes dans la théorie des nombres premiers.

Le développement du binôme $(1-x)^m$ donne ainsi, pour m quelconque, les séries

$$(109)\quad \begin{cases} \dfrac{V_{mr}}{V_r^m} = V_0 - \dfrac{m}{1}\dfrac{V_r}{V_r} + \dfrac{m(m-1)}{1.2}\dfrac{V_{2r}}{V_r^2} - \dfrac{m(m-1)(m-2)}{1.2.3}\dfrac{V_{3r}}{V_r^3} + \ldots, \\ \dfrac{U_{mr}}{V_r^m} = \dfrac{m}{1}\dfrac{U_r}{V_r} - \dfrac{m(m-1)}{1.2}\dfrac{U_{2r}}{V_r^2} + \dfrac{m(m-1)(m-2)}{1.2.3}\dfrac{U_{3r}}{V_r^3} - \ldots, \end{cases}$$

que l'on aurait pu déduire de la série de BERNOULLI; pour $m = -1$, on a

$$(110)\quad \begin{cases} \dfrac{V_r^2}{Q^r} = V_0 + \dfrac{V_r}{V_r} + \dfrac{V_{2r}}{V_r^2} + \dfrac{V_{3r}}{V_r^3} + \ldots, \\ \dfrac{U_{2r}}{Q^r} = \dfrac{U_r}{V_r} + \dfrac{U_{2r}}{V_r^2} + \dfrac{U_{3r}}{V_r^3} + \dfrac{U_{4r}}{V_r^4} + \ldots, \end{cases}$$

et, par exemple, dans la série de FIBONACCI

$$(111)\quad \begin{cases} 9 = 2 + \dfrac{3}{3} + \dfrac{7}{9} + \dfrac{18}{27} + \dfrac{47}{81} + \ldots, \\ 3 = \dfrac{1}{3} + \dfrac{3}{9} + \dfrac{8}{27} + \dfrac{21}{81} + \dfrac{55}{243} + \ldots; \end{cases}$$

les numérateurs de ces deux séries de fractions sont donnés par la relation de récurrence

$$N_{n+2} = 3N_{n+1} - N_n.$$

On obtiendra des formules semblables pour $m = \pm \dfrac{1}{2}$; le développement de

$$(1+x)^m \pm (1-x)^m$$

donne des formules analogues aux relations (109).

Le développement de Log $(1-x)$ donne les formules

$$(112)\quad \begin{cases} \operatorname{Log} \dfrac{V_r^2}{Q^r} = 1 + \dfrac{1}{2}\dfrac{V_{2r}}{V_r^2} + \dfrac{1}{3}\dfrac{V_{3r}}{V_r^3} + \dfrac{1}{4}\dfrac{V_{4r}}{V_r^4} + \ldots \\ \operatorname{Log} \dfrac{b^{2r}}{Q^r} = 2\sqrt{\Delta}\left[\dfrac{U_r}{V_r} + \dfrac{1}{2}\dfrac{U_{2r}}{V_r^2} + \dfrac{1}{3}\dfrac{U_{3r}}{V_r} + \dfrac{1}{4}\dfrac{U_{4r}}{V_r^7} + \ldots\right]; \end{cases}$$

celui de Log $\dfrac{1-x}{1+x}$ donne

$$(113)\quad \operatorname{Log} \frac{b^{2r}}{Q^r} = 2\sqrt{\Delta}\left[\frac{U_r}{V_r} + \frac{1}{3}\frac{\Delta U_{3r}}{V_r^3} + \frac{1}{5}\frac{\Delta^2 U_{5r}}{V_r^5} + \frac{1}{7}\frac{\Delta^2 U_{7r}}{V_r^7} + \ldots\right],$$

et, dans la série de PELL

$$(114)\quad \sqrt{2}\operatorname{Log}(1+\sqrt{2}) = 1 + \frac{1}{2.3} + \frac{1}{2^2.5} + \frac{1}{2^3.7} + \frac{1}{2^5.9} + \ldots.$$

La formule

$$\frac{1}{2}\operatorname{Log}\frac{z+h}{z-h} = hz\left[\frac{1}{z^2-h^2} - \frac{2}{3}\frac{h^2}{(z^2-h^2)^2} + \frac{2.4}{3.5}\frac{h^4}{(z^2-h^2)^3} - \frac{2.4.6}{3.5.7}\frac{h^6}{(z^2-h^2)^4} + \ldots\right],$$

dans laquelle on suppose

$$z+h = a^r,\quad z-h = b^r,\quad z^2-h^2 = Q^r,\quad h^2 = \frac{\Delta U_r^2}{4},$$

donne encore

$$(115)\quad \operatorname{Log}\frac{a^{2r}}{Q^r} = \frac{\sqrt{\Delta}U_r}{2}\left[\frac{1}{Q^r} - \frac{2}{3}\frac{\Delta U_r^2}{2^2Q^{2r}} + \frac{2.4}{3.5}\frac{\Delta^2 U_r^4}{2^3Q^{3r}} - \frac{2.4.6}{3.5.7}\frac{\Delta^3 U_r^6}{2^4Q^{4r}} + \ldots\right];$$

pour que cette série soit convergente, on doit avoir $\Delta U_r^2 < 4Q^r$; on trouve ainsi, à la limite de convergence

$$(116)\quad \operatorname{Log}(1+\sqrt{2}) = \sqrt{2}\left[1 - \frac{2}{3} + \frac{2.4}{3.5} - \frac{2.4.6}{3.5.7} + \frac{2.4.6.8}{3.5.7.9} - \ldots\right].$$

Les développements de arc sin z et de (arc sin z)^2 donnent, de même,

$$(117)\quad \begin{cases} \text{Log}\,\dfrac{a^{2r}}{Q^r} = \dfrac{\sqrt{\Delta}\,U_r}{Q^{\frac{r}{2}}}\left[1 - \dfrac{1}{1.2.3}\,\dfrac{\Delta U_r^2}{2^2 Q^r} + \dfrac{(1.3)^2}{1.2.3.4.5}\,\dfrac{\Delta^2 U_r^4}{2^4 Q^{2r}} - \cdots\right], \\ \dfrac{1}{4}\,\text{Log}^2\,\dfrac{a^{2r}}{Q^r} = \dfrac{\Delta U_r^2}{2^2 Q^r} - \dfrac{1}{2}\cdot\dfrac{2}{3}\,\dfrac{\Delta^2 U_r^4}{2^4 Q^{2r}} + \dfrac{1}{3}\cdot\dfrac{2.4}{3.5}\,\dfrac{\Delta^3 U_r^6}{2^6 Q^{3r}} - \cdots\cdots, \end{cases}$$

et, à la limite de convergence,

$$(118)\quad \begin{cases} \text{Log}\,(1+\sqrt{2}) = 1 - \dfrac{1}{1.2.3} + \dfrac{(1.3)^2}{1.2.3.4.5} - \dfrac{(1.3.5)^2}{1.2.3.4.5.6.7} + \cdots, \\ \text{Log}^2\,(1+\sqrt{2}) = 1 - \dfrac{1}{2}\cdot\dfrac{2}{3} + \dfrac{1}{3}\cdot\dfrac{2.4}{3.5} - \dfrac{1}{5}\cdot\dfrac{2.4.6}{3.5.7} + \cdots \end{cases}$$

La formule remarquable de M. SCHOLTZ conduit au développement

$$(119)\quad \text{Log}^3\,\frac{a^{2r}}{Q^r} = \frac{\Delta^{\frac{3}{2}} U_r^3}{Q^{\frac{3r}{2}}}\left[1 - \frac{3.3}{4.5}\left(1+\frac{1}{3^2}\right)\frac{\Delta U_r^2}{2^2 Q^r} + \frac{3.5.3}{4.6.7}\left(1+\frac{1}{3^2}+\frac{1}{5^2}\right)\frac{\Delta^2 U_r^4}{2^4 Q^{2r}} - \cdots \right.$$

$$\left. \pm \frac{3.5.7\ldots(2n-1)\,3}{4.6.8\ldots 2n\,(2n+1)}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\frac{1}{(2n-1)^2}\right)\frac{\Delta^{n-1}\,U_r^{2n-2}}{2^{2n-2}Q^{(n-1)r}} \mp \cdots\right],$$

et, à la limite de convergence

$$(120)\quad \text{Log}^3\,(1+\sqrt{2}) = 1 - \frac{3.3}{4.5}\left(1+\frac{1}{3^2}\right) + \frac{3.5.3}{4.6.7}\left(1+\frac{1}{3^2}+\frac{1}{5^2}\right) - \cdots.$$

Si l'on développe, par la formule de LAGRANGE, l'une des racines a^r ou b^r de l'équation

$$z^2 - zV_r + Q^r = 0,$$

on trouve

$$(121)\quad \begin{cases} b^r = \dfrac{Q^r}{V_r} + \dfrac{Q^{2r}}{V_r^3} + \dfrac{4}{2}\,\dfrac{Q^{3r}}{V_r^5} + \dfrac{5.6}{2.3}\,\dfrac{Q^{4r}}{V_r^7} + \cdots\cdots, \\ \text{Log}\,b^r = \text{Log}\,\dfrac{Q^r}{V_r} + \dfrac{Q^r}{V_r^2} + \dfrac{3}{2}\,\dfrac{Q^{2r}}{V_r^4} + \dfrac{5.4}{2.3}\,\dfrac{Q^{3r}}{V_r^6} + \cdots, \\ \dfrac{1}{3}\,b^{2r} = \dfrac{Q^{2r}}{2V_r^2} + \dfrac{Q^{3r}}{V_r^4} + \dfrac{5}{2}\,\dfrac{Q^{4r}}{V_r^6} + \dfrac{7.6}{2.3}\,\dfrac{Q^{5r}}{V_r^8} + \cdots\cdots \end{cases}$$

Si l'on fait encore, par la formule de LAGRANGE, le développement de y^{-n} suivant les puissances de z, en désignant par y l'une des racines de l'équation

$$y = 2 + \frac{z}{y},$$

on obtient

$$\left(\frac{2}{1+\sqrt{1+z}}\right)^n = 1 - \frac{n}{1}\frac{z}{4} + \frac{n(n+3)}{1.2}\left(\frac{z}{4}\right)^2 - \frac{n(n+4)(n+5)}{1.2.3}\left(\frac{z}{4}\right)^3$$
$$+ \frac{n(n+5)(n+6)(n+7)}{1.2.3.4}\left(\frac{z}{4}\right)^4 - \ldots,$$

et, en posant

$$\frac{z}{4} = -\frac{Q^r}{V_r^2},$$

on a

$$(122) \qquad \frac{V_{nr}a^{nr}}{Q^{nr}} = 1 + \frac{n}{1}\frac{Q^r}{V_r^2} + \frac{n(n+3)}{1.2}\frac{Q^{2r}}{V_r^4} + \frac{n(n+4)(n+5)}{1.2.3}\frac{Q^{3r}}{V_r^6} + \ldots;$$

cette série est convergente pour $\frac{Q^r}{V_r^2} < 1$; elle contient la généralisation de la formule (84).

On a encore

$$b^r = \frac{V_r - \sqrt{V_r^2 - 4Q^r}}{2},$$

et, en développant le radical par la formule du binôme,

$$(123) \qquad b^r = \frac{1}{2}\frac{2Q^r}{V_r} + \frac{1}{2.4}\frac{2^3Q^{2r}}{V_r^3} + \frac{1.3}{2.4.6}\frac{2^5Q^{3r}}{V_r^5} + \ldots,$$

puis, à la limite de convergence,

$$(124) \qquad \sqrt{2} - 1 = \frac{1}{2} - \frac{1}{2.4} + \frac{1.3}{2.4.6} - \frac{1.3.5}{2.4.6.8} + \ldots$$

En appliquant la formule de BURMANN au développement de z suivant les puissances de $\frac{2z}{1+z^2}$, on obtiendrait, pour tout module de z inférieur à l'unité, et faisant ensuite $z = \frac{b^r}{a^r}$, la formule (121) donnée ci-dessus.

SECTION XIX.

Sur le calcul rapide des fractions continues périodiques.

On perfectionne, d'une manière notable, le calcul des réduites des fractions continues périodiques au moyen des formules suivantes. M. CATALAN a donné les relations :

$$\frac{z}{1-z^2}+\frac{z^2}{1-z^4}=\frac{z+z^2+z^3}{1-z^4},$$

$$\frac{z}{1-z^2}+\frac{z^2}{1-z^4}+\frac{z^4}{1-z^8}=\frac{z+z^2+z^3+z^4+z^5+z^6+z^7}{1-z^8},$$

$$\frac{z}{1-z^2}+\frac{z^2}{1-z^4}+\frac{z^4}{1-z^8}+\frac{z^8}{1-z^{16}}=\frac{z+z^2+\ldots\ldots+z^{14}+z^{15}}{1-z^{16}},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ;$$

on a, plus généralement,

$$\frac{z}{1-z^2}+\frac{z^2}{1-z^4}+\ldots+\frac{z^{2^{n-1}}}{1-z^{2^n}}=\frac{1}{1-z}\cdot\frac{z-z^{2^n}}{1-z^{2^n}};$$

par conséquent, si l'on fait $z=\dfrac{b^r}{a^r}$, on obtient la formule

$$(125)\qquad \frac{Q^r}{U_{2r}}+\frac{Q^{2r}}{U_{4r}}+\ldots\ldots+\frac{Q^{2^{n-1}.r}}{U_{2^n.r}}=\frac{Q^r\,U_{(2^n-1)r}}{U_r\,U_{2^n.r}}.$$

Lorsque n augmente indéfiniment, on a, pour les séries de première et de seconde espèce,

$$(126)\qquad \frac{b^r}{U_r}=\frac{Q^r}{U_{2r}}+\frac{Q^{2r}}{U_{4r}}+\frac{Q^{4r}}{U_{8r}}+\ldots\ldots$$

Par exemple, dans la série de FIBONACCI, pour $r=1$,

$$(127)\qquad \frac{1-\sqrt{5}}{2}=-\frac{1}{1}+\frac{1}{3}+\frac{1}{3.7}+\frac{1}{3.7.47}+\frac{1}{3.7.47.2207}+\ldots;$$

chacun des nouveaux facteurs des dénominateurs est égal au carré du précédent, diminué de deux unités; de même, dans la série de PELL,

$$(128)\qquad 1-\sqrt{2}=-\frac{1}{2}+\frac{1}{2^2.3}+\frac{1}{2^3.3.17}+\frac{1}{2^4.3.17.577}+\ldots;$$

chacun des nouveaux facteurs des dénominateurs est égal, par les formules de duplication, au double du carré du précédent, diminué de l'unité.

Ces développements sont très-rapidement convergents; c'est, en quelque sorte, la combinaison du calcul logarithmique et du calcul par les fractions continues. Ainsi le dénominateur de la trentième-deuxième fraction de la formule (127), est à peu près égal à

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{2^{32}},$$

et contient *deux-cent millions de chiffres*, environ ; pour écrire le dénominateur de la soixante-quatrième fraction de la formule (128), il faudrait plus de *deux-cent millions de siècles.*

Nous avons d'ailleurs démontré (Section XI), que les différents facteurs des dénominateurs sont premiers entre eux deux à deux, et contiennent, par conséquent, des facteurs premiers tous différents ; il en résulte que dans la somme des n premiers termes de ces séries, il n'y aura pas lieu de réduire cette somme à une plus simple expression. Nous montrerons, de plus, que tous ces facteurs, premiers et différents, appartiennent à des formes, linéaires et quadratiques, déterminées.

On a, plus généralement, l'identité

$$(129)\qquad \frac{z-z^q}{(1-z)(1-z^q)}+\frac{z^q-z^{pq}}{(1-z^q)(1-z^{pq})}=\frac{z-z^{pq}}{(1-z)(1-z^{pq})};$$

si l'on remplace q par p^n, on a donc

$$(130)\qquad \frac{z-z^{p^n}}{(1-z)(1-z^{p^n})}+\frac{z^{p^n}-z^{p^{n+1}}}{(1-z^{p^n})(1-z^{p^{n+1}})}=\frac{z-z^{p^{n+1}}}{(1-z)(1-z^{p^{n+1}})}.$$

Si l'on fait successivement n égal à 1, 2, 3, ... n, et si l'on ajoute les égalités obtenues, on a

$$(131)\qquad \frac{z-z^p}{(1-z)(1-z^p)}+\frac{z^p-z^{p^2}}{(1-z^p)(1-z^{p^2})}+\frac{z^{p^2}-z^{p^3}}{(1-z^{p^2})(1-z^{p^3})}+\dots$$
$$+\frac{z^{p^n}-z^{p^{n+1}}}{(1-z^{p^n})(1-z^{p^{n+1}})}=\frac{z-z^{p^{n+1}}}{(1-z)(1-z^{p^{n+1}})}.$$

Faisons maintenant $z=\dfrac{b^r}{a^r}$, nous obtenons la formule

$$(132)\qquad \frac{Q^r U_{(p-1)r}}{U_r U_{pr}}+\frac{Q^{pr} U_{(p-1)pr}}{U_{pr} U_{p^2r}}+\frac{Q^{p^2r} U_{(p-1)p^2r}}{U_{p^2r} U_{p^3r}}+\dots+\frac{Q^{p^nr} U_{(p-1)p^nr}}{U_{p^nr} U_{p^{n+1}r}}+\frac{Q^r U_{(p^{n+1}-1)r}}{U_r U_{p^{n+1}r}}.$$

On calculera d'ailleurs les numérateurs et les dénominateurs de ces fractions, au moyen des formules de multiplication des fonctions numériques que nous avons données. Si p désigne un nombre impair, on obtient une formule analogue en changeant U en V. On peut encore appliquer ces formules aux fonctions circulaires.

Nous donnerons plus tard les formules analogues que l'on déduit de la théorie des fonctions elliptiques, et, en particulier, les sommes des inverses des termes U_n et de leurs puissances semblables.

SECTION XX.

Des relations des fonctions U_n et V_n avec la théorie de l'équation binôme.

On sait, par la théorie de l'équation binôme, exposée dans la dernière section des *Disquisitiones Arithmeticæ*, que si p désigne un nombre premier impair, le quotient

$$4\,\frac{z^p - 1}{z - 1} = 4\,(z^{p-1} + z^{p-2} + z^{p-3} + \ldots + z^2 + z + 1)$$

peut être écrit sous la forme

$$4\,\frac{z^p - 1}{z - 1} = Y^2 \pm pZ^2,$$

dans laquelle Y et Z sont des polynomes en z à coefficients entiers; on prend le signe $+$ lorsque p désigne un nombre premier de la forme $4q + 3$, et le signe $-$, lorsque p désigne un nombre premier de la forme $4q + 1$. Si l'on fait dans cette formule $z = \sqrt{\frac{a^r}{b^r}}$, on en déduit successivement pour $p = 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots$, les résultats suivants:

$$(133)\quad \left\{\begin{aligned}
4\,\frac{U_{3r}}{U_r} &= \Delta U_r^2 + 3Q^r, \\
4\,\frac{U_{5r}}{U_r} &= [2V_{2r} + Q^r]^2 - 5Q^{2r}, \\
4\,\frac{U_{7r}}{U_r} &= \Delta\,[2U_{3r} + Q^r U_r]^2 + 7Q^{2r} V_r^2, \\
4\,\frac{U_{11r}}{U_r} &= \Delta\,[2U_{5r} + Q^r U_{3r} - 2Q^{2r} U_r]^2 + 11Q^{2r} V_{3r}^2, \\
4\,\frac{U_{13r}}{U_r} &= [2V_{6r} + Q^r V_{4r} + 4Q^{2r} V_{2r} - Q^{3r}]^2 - 13Q^{2r}\,[V_{4r} + Q^{2r}]^2, \\
4\,\frac{U_{17r}}{U_r} &= [2V_{8r} + Q^r V_{6r} + 5Q^{2r} V_{4r} + 7Q^{3r} V_{2r} + 4Q^{4r}]^2 \\
&\qquad - 17Q^{2r}\,[V_{6r} + Q^r V_{4r} + Q^{2r} V_{2r} + 2Q^{3r}]^2, \\
4\,\frac{U_{19r}}{U_r} &= \Delta\,[2U_{9r} + Q^r U_{7r} - 4Q^{2r} U_{5r} + 3Q^{3r} U_{3r} + 5Q^{4r} U_r]^2 \\
&\qquad + 19Q^{2r}\,[V_{9r} + Q^r V_{7r} - Q^{3r} V_{3r} - 2Q^{4r} V_r]^2, \\
4\,\frac{U_{23r}}{U_r} &= \Delta\,[2U_{11r} + Q^r U_{9r} - 5Q^{2r} U_{7r} - 8Q^{3r} U_{5r} - 7Q^{4r} U_{3r} - 4Q^{5r} U_r]^2 \\
&\qquad + 23Q^{2r}\,[V_{9r} + Q^r V_{7r} - Q^{3r} V_{3r} - 2Q^{4r} V_r]^2, \\
4\,\frac{U_{29r}}{U_r} &= [2V_{14r} + Q^r V_{12r} + 8Q^{2r} V_{10r} - 3Q^{3r} V_{8r} + Q^{4r} V_{6r} - 2Q^{5r} V_{4r} + 3Q^{6r} V_{2r} \\
&\qquad + 9Q^{7r}]^2 - 29Q^{2r}\,[V_{12r} + Q^{2r} V_{8r} - Q^{3r} V_{6r} + Q^{5r}{}_{2r} V + Q^{6r}]^2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right.$$

On a, par conséquent, la proposition suivante :

THÉORÈME : *Si p désigne un nombre premier de la forme $4q+1$, le quotient $4\frac{U_{pr}}{U_r}$ peut se mettre sous la forme $Y^2 - pZ^2$, et si p désigne un nombre premier de la forme $4q+3$, le quotient $4\frac{U_{pr}}{U_r}$ peut se mettre sont la forme $\Delta Y^2 + pZ^2$.*

D'ailleurs, en changeant z en $-z$, on obtiendra un résultat semblable pour le quotient $4\frac{V_{pr}}{V_r}$. On généralise ainsi un théorème donné par LEGENDRE, et dont la démonstration est, de cette façon, rendue plus simple. Il résulte encore des formules (133) une autre conséquence importante. En effet, nous avons laissé jusqu'à présent Δ arbitraire; mais, s'il s'agit des fonctions de troisième espèce, nous pouvons supposer $-\Delta$ égal au produit d'un carré par un nombre premier p de la forme $4q+3$;* alors, on voit que les quotients $4\frac{U_{pr}}{pU_r}$ et $4\frac{V_{pr}}{pV_r}$ sont égaux à une différence de carrés, et, par suite, décomposables en un produit de deux facteurs. On a donc cette proposition :

THÉORÈME : *Si $-\Delta$ est égal au produit d'un nombre premier p de la forme $4q+3$ par un carré, les quotients $4\frac{U_{pr}}{pU_r}$ et $4\frac{V_{pr}}{pV_r}$ sont, quelle que soit la valeur entière de r, décomposables en un produit de deux facteurs entiers.*

Si nous considérons l'équation fondamentale

$$x^2 = x - 2$$

dans laquelle $\Delta = -7$, nous obtenons, par exemple,

$$U_{11} = +23, \quad U_{77} = -26\ 4721893121\ ;$$

et, par suite,

$$U_{77} = -7 \times 23 \times 11087 \times 148303.$$

Nous démontrerons plus loin que les diviseurs premiers de $\frac{4U_{77}}{7U_{11}}$ appartiennent aux formes linéaires $77q \pm 1$; par conséquent, le nombre 11087 est premier, sans qu'il soit nécessaire d'essayer ces diviseurs, puisque le premier des nombres de la forme linéaire indiquée, est supérieur à la racine carrée de 11087; pour le facteur 148303 il n'y a que le diviseur 307 à essayer. On a encore, dans la même série

$$U_{13} = -1, \quad U_{91} = -384\ 17168\ 38057,$$

* En effet, il suffit de déterminer Q par la relation $4Q - P^2 = pK^2$.

et, par suite,

$$U_{91} = -7 \times 712711 \times 770041.$$

Ces deux derniers facteurs sont premiers; il n'y a que deux diviseurs à essayer. On comprend ainsi comment il est possible d'appliquer le théorème précédent, à la recherche directe de très-grands nombres premiers, par la considération des séries de troisième espèce.

SECTION XXI.

Sur les congruences du Triangle Arithmétique de PASCAL, *et sur une généralisation du théorème de* FERMAT.

En désignant par C_m^n le nombre des combinaisons de m objets pris n à n, on a les deux formules fondamentales

$$C_m^n = \frac{m(m-1)\ldots(m-n+1)}{1.2.3\ldots n}$$

$$C_m^n = C_{m-1}^n + C_{m-1}^{n-1};$$

par conséquent, lorsque p est premier, on a pour n entier compris entre 0 et p, la congruence

(134) $$C_p^n \equiv 0, \quad (\text{Mod. } p);$$

pour n compris entre 0 et $p-1$,

(135) $$C_{p-1}^n \equiv (-1)^n, \quad (\text{Mod. } p);$$

pour n compris entre 1 et p

(136) $$C_{p+1}^n \equiv 0, \quad (\text{Mod. } p).$$

En d'autres termes, dans le triangle arithmétique de PASCAL, tous les nombres de la $p^{\text{ième}}$ ligne sont, pour p premier, divisibles par p, à l'exception des coefficients extrêmes égaux à l'unité; les coefficients de la $(p-1)^{\text{ième}}$ ligne donnent alternativement pour résidus $+1$ et -1; ceux de la $(p+1)^{\text{ième}}$ ligne sont divisibles par p, en exceptant les quatre coefficients extrêmes, égaux à l'unité.

Si l'on continue la formation du triangle arithmétique, en ne conservant que les résidus suivant le module p, on reforme deux fois le triangle arithmétique des (p — 1) premières lignes; puis, à partir de la $(2p)^{\text{ième}}$ ligne, on le reforme trois fois; mais les résidus du triangle intermédiaire sont multipliés par 2; à partir de la $(3p)^{\text{ième}}$ ligne, le triangle des résidus est reproduit quatre

fois, mais les nombres de ces triangles sont respectivement multipliés par les coefficients 1, 3, 3, 1 de la troisième puissance du binôme, et ainsi de suite.

On a donc, en général,

$$C_m^n \equiv C_{m_1}^{n_1} \times C_\mu^\nu, \quad (\text{Mod. } p),$$

m_1 et n_1 désignant les entiers de $\frac{m}{p}$ et de $\frac{n}{p}$, et μ et ν les résidus de m et de n. On a, de même

$$C_{m_1}^{n_1} \equiv C_{m_2}^{n_2} \times C_{\mu_1}^{\nu_1}, \quad (\text{Mod. } p),$$

et, par suite,

(137) $$C_m^n \equiv C_{\mu_1}^{\nu_1} \times C_{\mu_2}^{\nu_2} \times C_{\mu_3}^{\nu_3} \times \ldots, \quad (\text{Mod. } p),$$

$\mu_1, \mu_2, \mu_3, \ldots$ désignant les résidus de m et des entiers de $\frac{m}{p}, \frac{m}{p^2}, \frac{m}{p^3}, \ldots$, et de même pour $\nu_1, \nu_2, \nu_3, \ldots$.

Par conséquent, si l'on veut trouver le reste de la division de C_m^n par un nombre premier, il suffit d'appliquer la formule précédente, jusqu'à ce qu'on ait ramené les deux indices de C, à des nombres inférieurs à p.

Nous venons de voir que les coefficients de la puissance p du binôme sont entiers et divisibles par p, lorsque p désigne un nombre premier, en exceptant toutefois les coefficients des puissances $p^{\text{ièmes}}$. En désignant par $\alpha, \beta, \gamma, \ldots. \lambda$, des entiers quelconques, en nombre n, on a donc

$$[\alpha + \beta + \gamma + \ldots. + \lambda]^p - [\alpha^p + \beta^p + \gamma^p + \ldots. + \lambda^p] \equiv 0, \quad (\text{Mod. } p),$$

et, pour $\alpha = \beta = \gamma = \ldots. = \lambda = 1$, on obtient

$$n^p - n \equiv 0, \quad (\text{Mod. } p).$$

C'est dans cette congruence que consiste le théorème de FERMAT, que l'on peut généraliser de la manière suivante, différente de celle que l'on doit à EULER. Si $\alpha, \beta, \gamma, \ldots \lambda$, désignent les puissances $q^{\text{ièmes}}$ des racines d'une équation à coefficients entiers, et S_q leur somme, le premier membre de la congruence précédente représente le produit par p d'une fonction symétrique, entière et à coefficients entiers, des racines, et, par conséquent, des coefficients de l'équation proposée. On a donc

$$S_{pq} \equiv S_q^p, \quad (\text{Mod. } p),$$

et, par l'application du théorème de FERMAT,

(138) $$S_{pq} \equiv S_q, \quad (\text{Mod. } p).$$

L'étude des diviseurs premiers de la fonction numérique S_n et de quelques autres analogues est trés-importante; on a, en particulier, pour $n = 1$ et $S_1 = 0$, comme dans l'équation

$$x^3 = x + 1,$$

la congruence

$$S_p \equiv 0\,, \quad (\text{Mod. } p)\,;$$

on en déduit inversement que *si, dans le cas de $S_1 = 0$, on a S_n divisible par p, pour $n = p$, et non auparavant, le nombre p est un nombre premier.* En effet, supposons p égal, par exemple, au produit de deux nombres premiers g et h. On a

$$S_{gh} \equiv S_h\,, \quad (\text{Mod. } g)$$
$$S_{gh} \equiv S_g\,, \quad (\text{Mod. } h)\,;$$

par conséquent, si l'on a trouvé

$$S_{gh} \equiv 0\,, \quad (\text{Mod. } gh)\,,$$

on aura aussi

$$S_g \equiv 0\,, \quad (\text{Mod. } h)\,,$$
$$S_h \equiv 0\,, \quad (\text{Mod. } g)\,,$$

et, par le théorème démontré,

$$S_g \equiv S_h \equiv 0\,, \quad (\text{Mod. } gh)\,.$$

Ainsi S_{gh} ne serait pas le premier des nombres S_n divisible par gh.

On peut obtenir, de cette façon, un grand nombre de théorèmes servant, comme celui de WILSON, à vérifier les nombres premiers. Nous laisserons de côté, pour l'instant, les développements curieux et nouveaux que nous avons ainsi trouvés, pour ne considérer que ceux que l'on tire des fonctions numériques simplement périodiques.

SECTION XXII.

Sur la théorie des nombres premiers dans leurs rapports avec les progressions arithmétiques.

La doctrine des nombres premiers a été ébauchée par EUCLIDE et ERATOSTHÈNE. On doit à EUCLIDE la théorie des diviseurs et des multiples communs de deux ou plusieurs nombres donnés, la représentation des nombres composés à l'aide de leurs facteurs, et la démonstration de l'infinité des nombres premiers, que l'on peut étendre facilement à la preuve de l'infinité des nombres premiers appartenant aux formes linéaires $4x + 3$ et $6x + 5$. Nous donnerons, dans la Section XXIV, une démonstration élémentaire concernant l'infinité des nombres premiers de la forme $mx + 1$, quelle que soit la valeur de m. On sait d'ailleurs que, par l'emploi des séries infinies, LEJEUNE-DIRICHLET est parvenu à démontrer l'infinité des nombres premiers de la

forme linéaire $a + bx$, dans laquelle a et b sont deux entiers quelconques premiers entre eux.*

On doit à ERATOSTHÈNE une méthode ingénieuse connue sous le nom de *Crible Arithmétique*, qui conduit à la formation de la table des nombres premiers et des nombres composés; on possède, depuis les travaux de CHERNAC, de BURCKHARDT et de DASE, la table des neuf premiers millions; LEBESGUE a indiqué un procédé qui permet de diminuer le volume de ces tables.† D'autre part, M. GLAISHER a évalué la multitude des nombres premiers compris dans ces tables, afin de comparer les formules théoriques données par GAUSS, LEGENDRE, TCHEBYCHEF et HEARGRAVE, pour exprimer la quantité des nombres premiers inférieurs à un entier donné. M. GLAISHER, en comptant 1 et 2 comme premiers, a trouvé les valeurs suivantes: ‡

pour le	premier	million,	78499	nombres	premiers,
"	deuxième	"	, 70433	"	,
"	troisième	"	, 67885	"	,
"	septième	"	, 63799	"	,
"	huitième	"	, 63158	"	,
"	neuvième	"	, 62760	"	.

Les principes d'EUCLIDE et d'ERATOSTHÈNE conduisent ainsi à une première méthode de vérification des nombres premiers, non compris dans les Tables, et de décomposition des nombres très-grands en leurs facteurs premiers, par l'essai successif de la division d'un nombre *fixe*, le nombre donné, par tous les nombres premiers inférieurs à sa racine carrée. Mais c'est là une méthode indirecte qui devient absolument impraticable, dès que le nombre donné a dix chiffres.

En suivant cette voie, M. DORMOY est arrivé par des considérations ingénieuses, déduites de la théorie de certains nombres, qu'il a appelés *objectifs* (et dans lesquels on retrouve sous le nom *d'objectifs de l'unité* les différents termes de la série de FIBONACCI), à l'établissement d'une formule générale de nombres premiers. Malheureusement, même pour des limites peu élevées, cette formule contient des coefficients considérables qui en rendent l'application illusoire. ||

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* *Abhandlungen der Berliner Akademie*, Berlin, 1837.

† CHERNAC.—*Cribrum Arithmeticum* de 1 à 1020000. Deventer, 1811.

BURCKHARDT.—*Tables des diviseurs* jusqu'à 3036000. Paris, 1814–1817.

DASE.—*Factoren Tafeln* de 6000000 à 9000000. Vienne, 1862–1865.

LEBESGUE.—*Tables diverses pour la décomposition des nombres en leurs facteurs premiers.* Paris, 1864.

‡ *Preliminary accounts of the results of an enumeration of the primes in Dase's and Burckhardt's tables.* Cambridge, 1876–1877.

|| E. DORMOY.—*Formule générale des nombres premiers et Théorie des Objectifs.* Paris, 1867.

Les nombres premiers sont distribués fort irrégulièrement dans la suite des nombres entiers; c'est qu'en effet, d'une part, on voit que si μ désigne le plus petit multiple commun des nombres 2, 3, ... m, les nombres

$$\mu + 2, \ \mu + 3, \ldots, \mu + m,$$

sont respectivement divisibles par

$$2, \ 3, \ldots m.$$

Par conséquent, on peut toujours trouver m nombres consécutifs et composés, quelle que soit la valeur de m; mais, d'autre part, l'examen des tables permet de constater l'existence de deux nombres impairs consécutifs, très-grands, et premiers. M. GLAISHER a donné la liste des groupes, renfermés dans les tables, qui contiennent au moins cinquante nombres consécutifs et composés; ainsi, par exemple, les suivants:

111	nombres	composés et	consécutifs entre	370261	et	370373,
113	"	"	"	492113	et	492227,
131	"	"	"	1357201	et	1357333,
131	"	"	"	1561919	et	1562051,
147	"	"	"	2010733	et	2010881,

(*London Mathematical Society*, 10 Mai, 1877).

On sait encore démontrer qu'une fonction rationnelle de n

$$p = \phi(n),$$

ne peut continuellement donner des nombres premiers, puisque l'on a, quelque soit le nombre entier k,

$$\phi(n + kp) \equiv \phi(n), \quad (\text{Mod. } p),$$

c'est-à-dire que $\phi(n)$ est une fonction numérique périodique d'amplitude p. Il est donc fort difficile d'arriver à la loi de distribution des nombres premiers dans la série ordinaire des nombres entiers.

Cependant, il paraît naturel d'étudier les nombres premiers d'après leur loi de formation. L'étude approfondie de la méthode d'ERATOSTHÈNE a conduit le prince A. DE POLIGNAC, à d'intéressantes proprietés des *suites diatomiques;** à la même époque, M. TCHEBYCHEF, arrivait par des considérations peu différentes, à la démonstration de ce théorème remarquable: *Pour* $a > 3$, *il y a au moins un nombre premier compris entre a et* $2a - 2$.† On déduit immédiatement de là que le produit

$$1.2.3 \ldots n$$

* *Recherches nouvelles sur les nombres premiers;* par M. A. DE POLIGNAC; Paris, 1851. Il est curieux de constater que, sous le nom de *suite médiane*, on retrouve dans les séries diatomiques, les différents termes de la série de FERMAT.

† *Journal de Liouville*, t. XVII.

ne saurait être une puissance, ni un produit de puissances, ainsi que l'a montré M. LIOUVILLE. (*Journal de Liouville*, 2ᵉ série, t. II). En résumé, ces recherches sont basées sur la *considération des progressions arithmétiques.*

SECTION XXIII.

Sur la théorie des nombres premiers dans leurs rapports avec les progressions géométriques.

On doit à FERMAT des recherches profondes sur la théorie des nombres premiers, et basées sur la considération des *progressions géométriques.* C'est cette idée, distincte de la précédente, qui a donné naissance à la *théorie des résidus potentiels*, et, plus particulièrement, à celle des *résidus quadratiques.* De cette façon, on simplifie la vérification des nombres premiers très-grands, et diviseurs de la forme $a^n - 1$, ou plus généralement, de la forme $a^n - b^n$, pour a et b entiers, ainsi que la décomposition des nombres de cette forme en facteurs premiers. FERMAT avait remarqué la forme linéaire $nx + 1$ des diviseurs, et donné lui-même la décomposition de plusieurs termes de la série $2^n - 1$, et ainsi, celle du nombre $2^{37} - 1$, qu'il a trouvé divisible par 223 [*Lettre de* FERMAT, du 12 Octobre 1640].

M. GENOCCHI a remis dernièrement en lumière un curieux passage des œuvres du P. MERSENNE. Mais, pour en mieux saisir l'importance, nous rappellerons en quelques mots la théorie des *nombres parfaits.* On dit qu'un nombre est *parfait*, lorsque il est égal à la somme de ses parties *aliquotes*, c'est-à dire de tous ses diviseurs, excepté lui-même. En nous bornant au cas des nombres parfaits pairs, et en désignant par $b, c, \ldots.$ des nombres premiers différents, par $n = a^\alpha b^\beta c^\gamma d^\delta \ldots$, le nombre supposé parfait, on doit avoir

$$2^{\alpha+1} b^\beta c^\gamma \ldots = (1 + 2 + \ldots + 2^\alpha)(1 + b + b^2 + \ldots + b^\beta)(1 + c + c^2 + \ldots + c^\gamma) \ldots,$$

ou bien

$$b^\beta c^\gamma \ldots + \frac{b^\beta c^\gamma \ldots}{2^{\alpha+1} - 1} = (1 + b + b^2 + \ldots + b^\beta)(1 + c + c^2 + \ldots + c^\gamma) \ldots. ;$$

le second terme du premier membre est donc entier, et devient, après la division, de la forme $b^{\beta'} c^{\gamma'} \ldots$; mais d'autre part, le second membre qui contient un nombre de termes

$$\mu = (\beta + 1)(\gamma + 1) \ldots.,$$

doit se réduire aux deux termes du premier membre; par suite $\mu = 2$, $\beta = 1$, $\gamma = \delta = \ldots. = 0$; donc $n = 2^\alpha b$, et b est premier. Ainsi, les

nombres parfaits pairs appartiennent à la forme $n = 2^a b$, dans laquelle b doit être premier; on a d'ailleurs aisément, avec cette condition

$$b = 2^{a+1} - 1.$$

En résumé, il n'y a pas d'autres nombres parfaits pairs que les nombres

$$2^a(2^{a+1} - 1),$$

dans lesquels le second facteur est un nombre premier. Cette règle était connue d'EUCLIDE; mais ce géométre ne savait pas démontrer que l'on obtenait ainsi tous les nombres parfaits pairs, sans exception.

Voici maintenant le passage des Œuvres de MERSENNE:

"XIX. Ad ea quæ de Numeris ad calcem prop. 20. de Ballist. & puncto 14 Præfationis ad Hydraul. dicta sunt, adde inuentam artem quâ numeri, quotquot volueris, reperiantur qui cum suis partibus aliquotis in vnicam summam redactis, non solum duplam rationem habeant, (quales sunt 120, minimus omnium, 672, 523776, 1476304896, & 459818240, qui ductus in 3, numerum efficit 1379454720, cuius partes aliquotæ triplæ sunt, quales etiam sequentes 30240, 32760, 23569920, & alij infiniti, de quibus videatur Harmonia nostra, in qua 14182439040, & alij suarum partium aliquotarum subquadrupli) sed etiam sint in ratione data cum suis partibus aliquotis.

"Sunt etiam alij numeri, quos vocant amicabiles, quod habeant partes aliquotas à quibus mutuò reficiantur, quales sunt omnium minimi 220, & 284; huius enim aliquotæ partes illum efficiunt, vicéque versa partes illius aliquotæ hunc perfectè restituunt. Quales & 18416 & 17296; nec non 9437036, & 4363584 reperies, aliosque innumeros.

"Vbi fuerit operæ pretium aduertere XXVIII numeros à Petro Bungo pro perfectis exhibitos, capite XXVIII, libri de Numeris, non esse omnes Perfectos, quippe 20 sunt imperfecti, adeo vt solos octo perfectos habeat videlicet 6. 28. 496. 8128. 33550336. 8589869056. 137438691328, & 2305843008139952128; qui sunt è regione tabulae Bungi, 1, 2, 3, 4, 8, 10, 12, & 29: quique soli perfecti sunt, vt qui Bungum habuerint, errori medicinam faciant.

"Porrò numeri perfecti adeo rari sunt, vt vndecim dumtaxat potuerint hactenus inueniri: hoc est, alii tres à Bougianis differentes: neque enim vllus est alius perfectus ab illis octo, nisi superes exponentem numerum 62, progressionis duplæ ab 1 incipientis. Nonus enim perfectus est potestas exponentis 68 minus 1. Decimus, potestas exponentis 128, minus 1. Vndecimus denique, potestas 258, minus 1, hoc est potestas 257, vnitate decurtata, multiplicata per potestatem 256.

"Qui vndecim alios repererit, nouerit se analysim omnem, quæ fuerit hactenus, superasse: memineritque interea nullum esse perfectum à 17000 potestate ad 32000; & nullum potestatum interuallum tantum assignari posse, quin detur illud absque perfectis. Verbi gratia, si fuerit exponens 1050000, nullus erit numerus progressionis duplæ vsque ad 2090000, qui perfectis numeris serviat, hoc est qui minor vnitate, primus existat.

"Vnde clarum est quàm rari sint perfecti numeri, & quàm meritò viris perfectis comparentur; esseque vnam ex maximis totius Matheseos difficultatibus, præscriptam numerorum perfectorum multitudinum exhibere; quemadmodum & agnoscere num dati numeri 15, aut 20 caracteribus constantes, sint primi necne, cùm nequidem sæculum integrum huic examini, quocumque modo hactenus cognito, sufficiat." *

* F. MARINI MERSENNI MINIMI, COGITATA PHYSICO-MATHEMATICA. *In quibus tam naturæ quàm artis effectus admirandi certissimis demonstrationibus explicantur.* Paris, 1644. f° 11. de la Préface.

D'après ce passage, le tableau des nombres parfaits pairs serait le suivant:

Premier nombre parfait		$2\,(2^2-1)$,	Deuxième nombre parfait		$2^2\,(2^3-1)$,
Troisième	"	$2^4\,(2^5-1)$,	Quatrième	"	$2^6\,(2^7-1)$,
Cinquième	"	$2^{12}\,(2^{13}-1)$,	Sixième	"	$2^{16}\,(2^{17}-1)$,
Septième	"	$2^{18}\,(2^{19}-1)$,	Huitième	"	$2^{30}\,(2^{31}-1)$,
Neuvième	"	$2^{66}\,(2^{67}-1)$,	Dixième	"	$2^{126}\,(2^{127}-1)$,
Onzième	"	$2^{256}\,(2^{257}-1)$,	. . . .		

Ce passage est d'ailleurs rapporté dans un mémoire de C. N. WINSHEIM, inésré dans les *Novi Commentarii Academiæ Petropolitanæ, ad annum* MDCCXLIX (tom. II, pag. 78), et précédé des réflexions suivantes:

"Suspicio enim adesse videtur, utrum numerus nonus, perfecti locum tueri possit, quoniam ab acutissimo Mersenno exclusus reperitur, qui ejus in locum potestatem binarii $(2^{67}-1)\,2^{66}$ sive numerum decimum nonum perfectum Hanschii 1|47573|95258|96764|12927, substituit: digna certe mihi visa sunt verba viri perspicacissimi, ut hic integra exhibeantur."

Ainsi MERSENNE aurait démontré que, pour n compris entre 31 et 257, il n'existe pas de nombres premiers de la forme 2^n-1, en exceptant ceux pour lesquels n a pour valeur l'un des nombres

$$31,\ 67,\ 127,\ 257.$$

La preuve de non-décomposition du premier de ces nombres, $2^{31}-1$, n'a été donnée que plus tard, par EULER. En outre, M. F. LANDRY, au moyen d'une méthode inédite, et probablement fort simple, est parvenu à la décomposition de certains grands nombres en leurs facteurs premiers; il a, en effet, donné la décomposition des nombres

$$2^{41}-1,\quad 2^{43}-1,\quad 2^{47}-1,\quad 2^{53}-1,\quad 2^{59}-1,$$

en leurs facteurs premiers. De plus, on trouvé que $2^{73}-1, 2^{79}-1$ et $2^{113}-1$, sont respectivement divisibles par 439, 2687 et 3391. Enfin, on a le théorème suivant:

THÉORÈME: *Si* $4q+3$ *et* $8q+7$ *sont des nombres premiers, le nombre* $2^{4q+3}-1$ *est divisible par* $8q+7$.

En effet, d'après le thoérème de FERMAT, on a

$$2^{8q+6}-1\equiv 0,\quad (\text{Mod. } 8q+7),$$

et, par suite l'un des deux facteurs $2^{4q+3}+1$ ou $2^{4q+3}-1$ du premier membre de la congruence est divisible par le module; mais, d'autre part, on sait que 2 est résidu quadratique de tous les nombres premiers de l'une des formes $8n+1$ et $8n+7$; par conséquent, on a

$$2^{4q+3}-1\equiv 0,\quad (\text{Mod. } 8q+7);$$

en consultant la table des nombres premiers, on en conclut que pour les valeurs de n successivement égales à

$$11, 23, 83, 131, 179, 191, 239, 251, 359, 419, 431, 443, 491,$$

les nombres $2^n - 1$ sont respectivement divisibles par les facteurs

$$23, 47, 167, 263, 359, 383, 479, 503, 719, 839, 863, 887, 983.$$

Il résulte de ces diverses considérations que MERSENNE était en possession d'une méthode arithmétique qui ne nous est point parvenue. Cependant, il paraît naturel de penser que cette méthode ne devait pas s'éloigner des principes de FERMAT, et par conséquent, ne pas différer essentiellement de celle que nous déduirerons, plus loin, de l'inversion du théorème de FERMAT. Nous indiquons, en effet, comment il est possible d'arriver rapidement à l'étude du mode de composition des grands nombres dont il est parlé plus haut.

Nous donnons dans le tableau suivant, la décomposition des nombres U_n et V_n de la série de FERMAT, pour toutes les valeurs de n jusqu'à 64. Parmi les grands nombres premiers de ce tableau, on remarquera

1°. Cinq nombres de dix chiffres

42782 55361	facteur de	$2^{40} + 1$,
88314 18697	"	$2^{41} + 1$,
29315 42417	"	$2^{44} + 1$,
18247 26041	"	$2^{59} + 1$,
45622 84561	"	$2^{60} + 1$;

2°. Deux nombres de onze chiffres

5 44109 72897	facteur de	$2^{56} + 1$,
7 71586 73929	"	$2^{63} + 1$;

3°. Un nombre de douze chiffres

16 576[illegible]5 37521	facteur de	$2^{47} + 1$;

4°. Quatre nombres de treize chiffres

293 20310 07403	facteur de	$2^{43} + 1$,
443 26767 98593	"	$2^{49} - 1$,
436 39531 27297	"	$2^{49} + 1$,
320 34317 80337	"	$2^{59} - 1$;

5°. Un nombre de quatorze chiffres

$$2805\ 98107\ 62433 \quad \text{facteur de} \quad 2^{53}+1.$$

Il reste à déterminer la nature des trois nombres $2^{61}-1$, $\frac{1}{3}(2^{61}+1)$, et $2^{64}+1$. M. LANDRY pense que ces nombres sont premiers; mais, d'autre part, d'après MERSENNE, le premier de ces nombres serait composé; de plus, par la considération de calculs que j'ai effectués, et dont la théorie est indiquée plus loin, le dernier de ces nombres serait aussi composé. Il n'y a donc pas lieu de se prononcer pour le moment.

En dehors des décompositions renfermées dans le tableau, M. LANDRY a encore obtenu les diviseurs propres d'un certain nombre d'autres termes de cette série, à savoir

Pour $2^{65}+1$	4 09891 et 76 23851,	
$2^{69}+1$	16 87499 65921,	(premier),
$2^{75}-1$	1 00801 et 105 67201,	
$2^{75}+1$	113 38367 30401,	(premier),
$2^{105}+1$	6 64441 et 15 64921.	

De son côté, M. LE LASSEUR est parvenu aux mêmes résultats; mais, il a, en outre, indiqué l'identité

$$2^{4n+2}+1=(2^{2n+1}+2^{n+1}+1)(2^{2n+1}-2^{n+1}+1),$$

qui permet d'abréger les calculs. Cette identité, fort importante, sera généralisée ultérieurement.

TABLEAU DES FACTEURS PREMIERS DE LA SÉRIE RÉCURRENTE DE FERMAT.

d'après M. F. LANDRY.

U_n	Diviseurs de U_n		Valeurs de 2^n	V_n	Diviseurs de V_n
2^1-1	1	2^1	2	2^1+1	3
		2^2	4	2^2+1	5
2^3-1	7	2^3	8	2^3+1	3^2
		2^4	16	2^4+1	17
2^5-1	31	2^5	32	2^5+1	3.11
		2^6	64	2^6+1	5.13
2^7-1	127	2^7	128	2^7+1	3.43
		2^8	256	2^8+1	257
2^9-1	7.73	2^9	512	2^9+1	3^3.19
		2^{10}	1024	$2^{10}+1$	5^2.41
$2^{11}-1$	23.89	2^{11}	2048	$2^{11}+1$	3.683
		2^{12}	4096	$2^{12}+1$	17.241
$2^{13}-1$	8191	2^{13}	8192	$2^{13}+1$	3.2731
		2^{14}	16384	$2^{14}+1$	5.29.113
$2^{15}-1$	7.31.151	2^{15}	32768	$2^{15}+1$	3^2.11.331
		2^{16}	65536	$2^{16}+1$	65537
$2^{17}-1$	131071	2^{17}	131072	$2^{17}+1$	3.43691
		2^{18}	262144	$2^{18}+1$	5.13.37.109
$2^{19}-1$	524287	2^{19}	524288	$2^{19}+1$	3.174763
		2^{20}	1048576	$2^{20}+1$	17.61681
$2^{21}-1$	7^2.127.337	2^{21}	2097152	$2^{21}+1$	3^2.43.5419
		2^{22}	4194304	$2^{22}+1$	5.397.2113
$2^{23}-1$	47.178481	2^{23}	8388608	$2^{23}+1$	3.2796203
		2^{24}	16777216	$2^{24}+1$	97.257.673
$2^{25}-1$	31.601.1801	2^{25}	33554432	$2^{25}+1$	3.11.251.4051
		2^{26}	67108864	$2^{26}+1$	5.53.157.1613
$2^{27}-1$	7.73.262657	2^{27}	134217728	$2^{27}+1$	3^4.19.87211
		2^{28}	268435456	$2^{28}+1$	17.15790321
$2^{29}-1$	233.1103.2089	2^{29}	536870912	$2^{29}+1$	3.59.3033169
		2^{30}	1073741824	$2^{30}+1$	5^2.13.41.61.1321
$2^{31}-1$	2147483647	2^{31}	2147483648	$2^{31}+1$	3.715827883
		2^{32}	4294967296	$2^{32}+1$	641.6700417

TABLEAU DES FACTEURS PREMIERS DE LA SÉRIE RÉCURRENTE DE FERMAT.

(*Suite.*)

U_n	Diviseurs de U_n		Valeurs de 2^n	V_n	Diviseurs de V_n
$2^{33}-1$	7.23.89.599479	2^{33}	8589934592	$2^{33}+1$	3^2.67.683.20857
		2^{34}	17179869184	$2^{34}+1$	5.137.953.26317
$2^{35}-1$	31.71.127.122921	2^{35}	34359738368	$2^{35}+1$	3.11.43.281.86171
		2^{36}	68719476736	$2^{36}+1$	17.241.433.38737
$2^{37}-1$	223.616318177	2^{37}	137438953472	$2^{37}+1$	3.1777.25781083
		2^{38}	274877906944	$2^{38}+1$	5.229.457.525313
$2^{39}-1$	7.79.8191.121369	2^{39}	549755813888	$2^{39}+1$	3^2.2731.22366891
		2^{40}	1099511627776	$2^{40}+1$	257.4278255361
$2^{41}-1$	13367.164511353	2^{41}	2199023255552	$2^{41}+1$	3.83.8831418697
		2^{42}	4398046511104	$2^{42}+1$	5.13.29.113.1429.14449
$2^{43}-1$	431.9719.2099863	2^{43}	8796093022208	$2^{43}+1$	3.2932031007403
		2^{44}	17592186044416	$2^{44}+1$	17.353.2931542417
$2^{45}-1$	7.31.73.151.631.23311	2^{45}	35184372088832	$2^{45}+1$	3^3.11.19.331.18837001
		2^{46}	70368744177664	$2^{46}+1$	5.277.1013.1657.30269
$2^{47}-1$	2351.4513.13264529	2^{47}	140737488355328	$2^{47}+1$	3.283.165768537521
		2^{48}	281474976710656	$2^{48}+1$	193.65537.22253377
$2^{49}-1$	127.4432676798593	2^{49}	562949953421312	$2^{49}+1$	3.43.4363953127297
		2^{50}	1125899906842624	$2^{50}+1$	5^3.41.101.8101.268501
$2^{51}-1$	7.103.2143.11119.131071	2^{51}	2251799813685248	$2^{51}+1$	3^2.307.2857.6529 43691
		2^{52}	4503599627370496	$2^{52}+1$	17.858001.308761441
$2^{53}-1$	6361.69431.20394401	2^{53}	9007199254740992	$2^{53}+1$	3.107.28059810762433
		2^{54}	18014398509481984	$2^{54}+1$	5.13.37.109.246241.279073
$2^{55}-1$	23.31.89.881.3191.201961	2^{55}	36028797018963968	$2^{55}+1$	3.11^2 683.2971 48912491
		2^{56}	72057594037927936	$2^{56}+1$	257.5153.54410972897
$2^{57}-1$	7.32377.524287.1212847	2^{57}	144115188075855872	$2^{57}+1$	3^2.571.174763.160465489
		2^{58}	288230376151711744	$2^{58}+1$	5.107367629.536903681
$2^{59}-1$	179951.3203431780337	2^{59}	576460752303423488	$2^{59}+1$	3 2833.37171.1824726041
		2^{60}	1152921504606846976	$2^{60}+1$	17 241.61681.4562284561
$2^{61}-1$	. . . .	2^{61}	2305843009213693952	$2^{61}+1$	3. . . . .
		2^{62}	4611686018427387904	$2^{62}+1$	5.5581.8681.49477.384773
$2^{63}-1$	7^2.73.127.337.92737.649657	2^{63}	9223372036854775808	$2^{63}+1$	3^3.19.43.5419.77158673929
		2^{64}	18446744073709551616	$2^{64}+1$	. . . .

(Sera continué.)

THE ELASTIC ARCH.

By Henry T. Eddy, *Cincinnati, O.*

It is usual in the discussion of the mathematical principles of the inelastic arch, as an arch constructed of masonry is, in effect, to trace the curve of pressures due to the loading and to the thrust of the arch, and compare it with the configuration of the arch itself. This comparison shows, in the clearest manner, the stability or instability of the arch, and it enables the designer to form a ready opinion as to the effect of altering the shape of the arch, or of changing the loading, etc.

This curve of pressures is also, as is well known, the curve of equilibrium or catenary due to the loading and to the thrust. It is also frequently spoken of as the curve of moments, since it is well known that its ordinates multipled by the horizontal thrust are equal to the bending moments which would be caused by the given loading in an elastic girder on which no horizontal thrust acts, which girder is not necessarily straight

It is thus seen that the girder itself can be discussed by the help of an equilibrium polygon having any assumed thrust; and this is the process employed in treating the girder by the graphical method.

Now when we turn to the mathematical treatment of the elastic arch, or curved girder of any shape, acted on by a thrust, (which is the case when the reactions of its two supports are not parallel), we find that the discussions heretofore given have been based on analytic considerations exclusively, and the useful relations expressed by the equilibrium polygon are left out of view.

It is the object of this paper to point out the fundamental relationship existing between the elastic arch or crooked girder and the equilibrium curve due to its loading and to the thrust acting upon it. This relationship constitutes the basis of a complete graphical treatment of the elastic arch, subject to any possible conditions such as induced bending moments at its extremities or elsewhere, or the introduction of hinge joints at arbitrary points; and it affords at the same time a simple conception and interpretation of the analytic results arrived at in the usual investigation of the elastic arch.

It is evident that the bending moment at any point of an elastic arch is the algebraic sum of the moments of the forces and couples applied to the arch on either side of the assumed point.

Let us first consider the bending moment at any point of the arch which is due to a simple thrust alone. A simple thrust is induced in an arch by a variation of temperature or any other cause by which its natural span is made to differ from the actual distance between its points of support; but the bending moments due to a given thrust are identical whatever causes that thrust; whether it is a secondary effect due to the bending which the loading produces, or whether it is caused by some variation of temperature or position of its extremities which makes the natural span of the arch differ from the actual distance between its points of support. The word thrust is used to include the effects of contraction as well as elongation in the arch; the thrust in the former case being negative.

If the arch is not fixed in direction at the points of support, then the bending moment vanishes at these points. and the thrust acts along a line joining them, which is not necessarily horizontal. The bending moment at any point of the arch due to this thrust is the product of the thrust by its arm, which arm is the perpendicular distance from the assumed point to the line of the thrust. This product is evidently equal to the product of the horizontal component of the thrust by the vertical distance from the assumed point to the line of thrusts, as appears from an elementary consideration of the similarity of triangles.

Hence it appears that we have in this case arrived at the following important truth:

The neutral axis of an elastic arch is the equilibrium curve of the bending moments due to the thrust between its supports.

And this statement applies not only to an arch having hinge joints at its points of support, but to the elastic arch in general, as we now proceed to show.

Any case other than that already considered may be caused by the application of a couple at one or both of the points of support, thereby inducing a bending moment at one or both of these points. The effect of such a couple is not dependent upon the manner in which it is induced: it may be that it is arbitrarily applied, or it may be caused, as is usually the case, by the thrust: the bending moment, which a couple arbitrarily applied

at a point of support causes, is one which uniformly decreases from its point of application to the other point of support. Its effect is then to remove the thrust line from the point where the couple is applied to a new position, such that the vertical distance between the point of support, at which the couple is applied, and the new thrust line, multiplied by the thrust of the arch, is equal to the moment of the applied couple. The other extremity of the thrust line is unmoved by applying this couple. If in addition a second couple be applied at the second point of support the thrust line is removed from that point of support in a similar manner while the other extremity is unmoved.

But the same reasoning now applies, which we before employed, to find the moment at any point of the arch; it may be stated more explicitly thus:

The bending moment at any point of an elastic arch caused by the thrust, horizontal or inclined, is the product of the horizontal component of the thrust by the vertical ordinate between the point assumed on the neutral axis of the arch and the thrust line in its true position.

It may be noticed in this connection that it is always possible to determine the magnitude of the couples accompanying a simple thrust and applied at the points of support, from consideration of the conditions imposed on the arch by the amount of deflection horizontal or otherwise which is possible, but it does not fall within the scope of this paper to examine these conditions and show how to determine the couples accompanying a thrust.

Having considered the thrust and its equilibrium or moment curve, which is the neutral axis of the arch itself, let us in the second place consider the loading and its moment curve.

The bending moment at any assumed point of the arch due to the weights is the algebraic sum of the products obtained by multiplying each weight by its horizontal distance from the assumed point.

If the arch has hinge joints at the points of support, the loading can cause no bending moments at those points; but, if the arch is supported in some other way, it is evident that the moments due to the loading are accompanied by couples applied at the points of support, in the same manner as were the bending moments due to the thrust, and that the magnitude of these couples must be determined from the same considerations respecting deflection, etc., as determined those; indeed, each separate force which is applied to the arch, be it thrust, weight or any other force, causes bending moments throughout the arch which can be separately treated, and each must evidently

be treated in the same manner. We have, for convenience, grouped all the weights together. Hence we have reached another important truth:

In any elastic arch, the closing line of the moment curve, due to the weights alone, must be found from the same conditions and have the same relations to this moment curve that the thrust line has to the curved neutral axis of the arch.

It is to be noticed that the thrust caused by loading an arch induces bending moments of opposite sign from those induced by the loading itself, and that the bending moments really acting on the arch are the difference between those induced by the loading and those induced by the thrust. Hence appears the truth of the following statement, which combines together the separate effects of the thrust and the loading:

If that moment curve, due to the loading, which has for its horizontal thrust the thrust really acting in the arch, be superposed upon the curve of the arch itself, in such a manner that its closing line coincides with the thrust line of the arch, then the bending moment at any point of the arch is equal to the product of the horizontal thrust by the vertical ordinate between the assumed point and the moment curve due to the loading.

Hence the neutral axis of the arch may be considered to play the part of a curved closing line of the moment curve.

This proposition, respecting the coincidence of the thrust line and the closing line, affords the basis for a new graphical investigation of the elastic arch.*

The same principles may be applied to the elastic arch acted on by other than vertical forces.

*See "New Constructions in Graphical Statics." By Henry T. Eddy, published by D. Van Nostrand, New York, 1877.

RESEARCHES IN THE LUNAR THEORY.

By G. W. Hill, *Nyack Turnpike, N. Y.*

Chapter II.

(Continued from p. 147.)

The method of employing numerical values, from the outset, in the equations of condition, determining the a_i, is far less laborious than the literal development of these coefficients in powers of a parameter. For comparison with the results just given, we add the calculation of the coefficients by this method. The following table gives the numerical values of the symbols $[j, i]$, $[j]$ and (j), but the division by the quantity $2(4j^2 - 1) - 4m + m^2$ has been omitted; it is easier to perform this once for all at the end of the series of operations, than to divide each coefficient separately. Hence it must be understood that all the numbers in each department of the table are to be divided by the divisor which stands at the head of it.

Coefficients for a_1 and a_{-1}.

Divisor = 5.68314 08148 64695.

$[1] =$	$0.00861\ 47842\ 96261$	$[-1] = -$	$0.01178\ 75756\ 56865$
$(1) = -$	$0.00623\ 66553\ 18347$	$(-1) = -$	$0.04941\ 95042\ 02516$
$[1, -2] =$	$13.30665\ 60411$	$[-1, -3] = -$	$66.98979\ 68560$
$[1, -1] =$	$6\ 32993\ 22853$	$[-1, -2] = -$	$28.01307\ 31002$
$[1, 2] = -$	$10.71949\ 01593$	$[-1, 1] = -$	$10.96365\ 06556$
$[1, 3] = -$	$15.10904\ 80332$	$[-1, 2] = -$	$38.57409\ 27816$

Coefficients for a_2 and a_{-2}.

Divisor = 29.68314 08148 64695.

$2[2] =$	$0\ 00205\ 43632\ 76229$	$2[-2] = -$	$0.01834\ 79966\ 76898$
$2(2) = -$	$0.02909\ 07097\ 39048$	$2(-2) = -$	$0.07227\ 35586\ 23216$
$[2, -2] =$	$14.97672\ 37558$	$[-2, -4] = -$	$108.69586\ 45706$
$[2, -1] =$	$9.32666\ 40103$	$[-2, -3] = -$	$63.00980\ 48251$
$[2, 1] = -$	$13.00326\ 82750\ 49$	$[-2, -1] = -$	$8.67987\ 25398$

[2, 3] = — 50.03961 76194	[—2, 1] = — 3.64352 31954
[2, 4] = — 74.07269 86888	[—2, 2] = — 19.61044 21261

Coefficients for a_3 and a_{-3}.

Divisor = 69.68314 08149.

[3] = — 0.00113 35729 26473	[—3] = — 0.00793 43596
(3) = — 0.01768 33677	(—3) = — 0.03207 76506 69434
[3, —1] = 12.99224 12519	[—3, —4] = — 114.67538 20668
[3, 1] = — 18.10997 74284	[—3, —2] = — 35.57316 33864
[3, 2] = — 41.33769 10334	[—3, —1] = — 12.34544 97815
[3, 4] = — 103.14632 67728	[—3, 1] = 1.46318 59580

Coefficients for a_4 and a_{-4}.

Divisor = 125.68314 08.

2 [4] = — 0.00428 9733	2 [—4] = — 0.01449 0913
2 (4) = — 0.03864 29156	2 (—4) = — 0.06023 435
[4, —1] = 16.82502 987	[—4, —5] = — 182.50817 069
[4, 1] = — 22.66333 2	[—4, —3] = — 79.01980 9
[4, 2] = — 51.16496 6	[—4, —2] = — 42 51817 5
[4, 3] = — 85.50490 2	[—4, —1] = — 16 17823 8
[4, 5] = — 171.69968 135	[—4, 1] = 6.01654 053

Coefficients for a_5 and a_{-5}.

Divisor = 197.68314.

[5] = — 0.00272 9536	(5) = — 0.02896 299
[5, 1] = — 26.99534 4	[—5, —4] = — 138.68780 0
[5, 2] = — 60.26133 2	[—5, —3] = — 89.42181 0
[5, 3] = — 99.79795 8	[—5, —2] = — 49.88518 4
[5, 4] = — 145.60523 2	[—5, —1] = — 20.07791 2

Coefficients for a_6 and a_{-6}.

Divisor = 285.68314.

2 [6] = — 0.00622 021	2 (—6) = — 0.05640 548
[6, 1] = — 31.21669	[—6, —5] = — 214.46646
[6, 2] = — 68.99224	[—6, —4] = — 152.69091
[6, 3] = — 113.32666	[—6, —3] = — 100.35648

$[6, 4] = -164.21995 \qquad [-6, -2] = -\ 57.46319$

$[6, 5] = -221.67212 \qquad [-6, -1] = -\ 24.01103\,.$

These numbers are arranged for carrying the precision to quantities of the 13th order inclusive, and to 15 places of decimals. The quantities $[j, i]$ can be tested by differences, if 0 and the divisor with the negative sign are inserted in the proper places in the series of numbers; for it is evident that the second differences should be constant.

The final results are given below, where, in order that the degree of convergence of this process may be appreciated, we have given the value arising from the first approximation, and then, separately, the corrections arising severally from the second and third approximations. It must be borne in mind that each of these terms is the numerical value, not of an infinite series, but of a rational function of m, and, consequently admits of being computed exact to the last decimal place employed, and, in fact, is here so computed. Hence any error there may be in these values of the a_i arises only from the neglect of the terms of the following approximations, which, in half the number of cases, are of the 14th order, and, in the other half, of the 16th order. It is safe to affirm that these cannot, in any case, exceed two units in the 15th decimal.

	a_1.	a_{-1}.
1st apx, term of 2d order,	$+0.00151\ 58491\ 71593$	$-0.00869\ 58084\ 99634$
2d " " 6th "	$-0.00000\ 01416\ 98831$	$+0.00000\ 00615\ 51932$
3d " " 10th "	$+0.00000\ 00000\ 06801$	$-0.00000\ 00000\ 13838$

$$\frac{a_1}{a_0} = +0.00151\ 57074\ 79563, \quad \frac{a_{-1}}{a_0} = -0.00869\ 57469\ 61540,$$

	a_2.	a_{-2}.
1st apx., term of 4th order,	$+0.00000\ 58793\ 35016$	$+0.00000\ 01636\ 69405$
2d " " 8th "	$-0.00000\ 00006\ 78490$	$+0.00000\ 00001\ 21088$
3d " " 12th "	$+0.00000\ 00000\ 00052$	$-0.00000\ 00000\ 00007$

$$\frac{a_2}{a_0} = +0.00000\ 58786\ 56578, \quad \frac{a_{-2}}{a_0} = +0.00000\ 01637\ 90486,$$

	a_3.	a_{-3}.
1st apx., term of 6th order,	$+0.00000\ 00300\ 35759$	$+0.00000\ 00024\ 60338$
2d " " 10th "	$-0.00000\ 00000\ 04128$	$+0.00000\ 00000\ 00055$

$$\frac{a_3}{a_0} = +0.00000\ 00300\ 31632, \quad \frac{a_{-3}}{a_0} = +0.00000\ 00024\ 60393,$$

	a_4.	a_{-4}.
1st apx., term of 8th order,	+0.00000 00001 75296	+0.00000 00000 12284
2d " " 12th "	−0.00000 00000 00028	0 00000 00000 00000

$$\frac{a_4}{a_0} = +0.00000\ 00001\ 75268,\quad \frac{a_{-4}}{a_0} = +0.00000\ 00000\ 12284,$$

Of the 10th order, $\frac{a_5}{a_0} = +0.00000\ 00000\ 01107,\quad \frac{a_{-5}}{a_0} = +0.00000\ 00000\ 00064,$

Of the 12th order, $\frac{a_6}{a_0} = +0.00000\ 00000\ 00007,\quad \frac{a_{-6}}{a_0} = +0.00000\ 00000\ 00000.$

These give the following numerical expression for the coordinates,

$$\begin{aligned} r\cos v = a_0 [1 &- 0.00718\ 00394\ 81977\ \cos 2\tau \\ &+ 0.00000\ 60424\ 47064\ \cos 4\tau \\ &+ 0.00000\ 00324\ 92024\ \cos 6\tau \\ &+ 0.00000\ 00001\ 87552\ \cos 8\tau \\ &+ 0.00000\ 00000\ 01171\ \cos 10\tau \\ &+ 0.00000\ 00000\ 00008\ \cos 12\tau], \\ r\sin v = a_0 [&\ 0.01021\ 14544\ 41102\ \sin 2\tau \\ &+ 0\ 00000\ 57148\ 66093\ \sin 4\tau \\ &+ 0.00000\ 00275\ 71239\ \sin 6\tau \\ &+ 0.00000\ 00001\ 62985\ \sin 8\tau \\ &+ 0.00000\ 00000\ 01042\ \sin 10\tau \\ &+ 0.00000\ 00000\ 00007\ \sin 12\tau]. \end{aligned}$$

On comparison of these values with those obtained from the series in $\mathfrak{m}$, the differences are found to be only some units in the 11th decimal.

The coefficients tend to diminish with some regularity as we advance towards higher orders. This is shown by the following scheme of the logarithms and their differences:

	Δ	Δ^2		Δ	Δ^2
n 97.8561			98.0091		
				−3.2521	
94.7812			94.7570		+9356
	−2.2694			2.3165	
92.5118		+307	92.4405		871
	2.2387			2.2294	
90.2731		341	90.2111		363
	2.2046			2.1931	
88.0685		237	88.0180		201
	2.1809			2.1730	
85.8876			85.8450		

For verification the following equations were computed,

$$\Sigma_i . [(2i+1+\mathrm{m})^2 + 2\mathrm{m}^2]\, \mathrm{a}_i . [\Sigma_i . \mathrm{a}_i]^2 = 1.17141\ 84591\ 84518\ \mathrm{a}_0^3\,,$$

$$\Sigma_i . (-1)^i (2i+1)(2i+1+\mathrm{m})\, \mathrm{a}_i . [\Sigma_i . (-1)^i \mathrm{a}_i]^2 = 1.17141\ 84591\ 84513\ \mathrm{a}_0^2\,.$$

The small difference between the numbers is explained by the fact that, in these formulæ, the quantities a_i are, when i is somewhat large, multiplied by large numbers; as, for instance, a_6 by 169. From the average of these two results we get

$$\mathrm{a}_0 = 0.99909\ 31419\ 75298 \left[\frac{\mu}{n^2}\right]^{\frac{1}{3}}.$$

In the investigations of succeeding chapters, the function $\frac{\varkappa}{r^3}$ plays an important part. Hence we will here derive its development as a periodic function of τ by the method of special values. By dividing the quadrant, with reference to τ, into 6 equal parts, we obtain the advantage that the sines or cosines of the multiples of 2τ are either rational or involve $\sqrt{3}$. The special values of the coordinates and of $\frac{\varkappa}{r^3}$, thence deduced, are

τ.	$\frac{r}{\mathrm{a}_0}\cos v$.	$\frac{r}{\mathrm{a}_0}\sin v$.	$\frac{\varkappa}{r^3}$.
0°	0.99282 60356 45842	0.00000 00000 00000	1.19699 57017 23421
15	0.99378 49245 37167	0.00511 07041 52675	1.19348 68051 03032
30	0.99640 69264 50272	0.00884 83280 32746	1.18399 66676 76716
45	0.99999 39577 40480	0.01021 14268 70906	1.17125 64904 33157
60	1.00358 70309 15127	0.00883 84298 76613	1.15876 77987 29687
75	1.00622 11177 22330	0.00510 08054 31947	1.14978 07679 95764
90	1.00718 60496 23406	0.00000 00000 00000	1.14652 34925 50570 .

From the numbers of the last column, by the known process, we deduce

$$\begin{aligned}\frac{\varkappa}{r^3} = {} & \ 1.17150\ 80211\ 79225 \\ & + 0.02523\ 36924\ 97860 \cos 2\tau \\ & + 0.00025\ 15533\ 50012 \cos 4\tau \\ & + 0.00000\ 24118\ 79799 \cos 6\tau \\ & + 0.00000\ 00226\ 05851 \cos 8\tau \\ & + 0.00000\ 00002\ 08750 \cos 10\tau \\ & + 0.00000\ 00000\ 01908 \cos 12\tau \\ & + 0.00000\ 00000\ 00017 \cos 14\tau\,.\end{aligned}$$

The last coefficient has been added from induction, after which it becomes necessary, as is plain, to subtract an equal quantity from the coefficient of $\cos 10\tau$. Writing the logarithms, as in the former case, we have, the last logarithm being supplied from estimation,

	Δ	Δ^2	Δ^3
98.4020			
	$-$ 2.0014		
96.4006		$-$ 168	
	2.0182		$+$ 68
94.3824		100	
	2.0282		36
92.3542		64	
	2.0346		20
90.3196		44	
	2.0390		10
88.2806		34	
	2.0424		
86.2382			

It will be noticed how much slower this series converges than those for the coordinates.

Any information regarding the motion of satellites having long periods of revolution about their primaries will doubtless be welcome, as the series given by previous investigators are inadequate for showing anything in this direction. Hence this chapter will be terminated by a table of the more salient properties of the class of satellites having the radius vector at a minimum in syzygies and at a maximum in quadratures. For this end I have selected, besides the earth's moon, taken for the sake of comparison, the moons of 10, 9, 8, . . . , 3 lunations in the periods of their primaries, and also what may be called the moon of maximum lunation, as, of the class of satellites under discussion, exhibiting the complete round of phases, it has the longest lunation.

In order that the table may be readily applicable to satellites accompanying any planet, the canonical linear and temporal units, that is those for which μ and n' are both unity, will be used.

From the foregoing methods we obtain :

$$\text{For } \mathrm{m} = \frac{1}{10};$$

$$r \cos v = \mathrm{a}\,[1 - 0.011230 \cos 2\tau + 0.000015 \cos 4\tau], \qquad r \sin v = \mathrm{a}\,[\;0.016102 \sin 2\tau + 0.000014 \sin 4\tau],$$

$$\log \mathrm{a} = 9.3051648.$$

For $\mathrm{m} = \frac{1}{9}$;

$$r \cos v = \mathrm{a}\,[1 - 0.014044 \cos 2\tau + 0.0000247 \cos 4\tau], \qquad r \sin v = \mathrm{a}\,[\,0.020232 \sin 2\tau + 0.0000230 \sin 4\tau],$$

$$\log \mathrm{a} = 9.3326467\,.$$

For $\mathrm{m} = \frac{1}{8}$;

$$r \cos v = \mathrm{a}\,[1 - 0.018061 \cos 2\tau + 0.0000421 \cos 4\tau + 0.00000057 \cos 6\tau], \qquad r \sin v = \mathrm{a}\,[\,0.026172 \sin 2\tau + 0.0000388 \sin 4\tau + 0.00000048 \sin 6\tau],$$

$$\log \mathrm{a} = 9.3630019\,.$$

For $\mathrm{m} = \frac{1}{7}$;

$$r \cos v = \mathrm{a}\,[1 - 0.02407886 \cos 2\tau + 0.00007760 \cos 4\tau + 0.00000141 \cos 6\tau + 0.000000025 \cos 8\tau], \qquad r \sin v = \mathrm{a}\,[\,0.03516059 \sin 2\tau + 0.00007063 \sin 4\tau + 0.00000118 \sin 6\tau + 0.000000022 \sin 8\tau],$$

$$\log \mathrm{a} = 9.3969048\,.$$

For $\mathrm{m} = \frac{1}{6}$;

$$r \cos v = \mathrm{a}\,[1 - 0.03368245 \cos 2\tau + 0.00015943 \cos 4\tau + 0.000004077 \cos 6\tau + 0.000000097 \cos 8\tau], \qquad r \sin v = \mathrm{a}\,[\,0.04968194 \sin 2\tau + 0.00014312 \sin 4\tau + 0.000003393 \sin 6\tau + 0.000000084 \sin 8\tau],$$

$$\log \mathrm{a} = 9.4352928\,.$$

For $\mathrm{m} = \frac{1}{5}$;

$$r \cos v = \mathrm{a}\,[1 - 0.05038803 \cos 2\tau + 0.00038127 \cos 4\tau + 0.000014686 \cos 6\tau + 0.000000505 \cos 8\tau], \qquad r \sin v = \mathrm{a}\,[\,0.07536021 \sin 2\tau + 0.00033582 \sin 4\tau + 0.000012168 \sin 6\tau + 0.000000438 \sin 8\tau],$$

$$\log \mathrm{a} = 9.4795445\,.$$

For $\mathrm{m} = \frac{1}{4}$;

$$r \cos v = \mathrm{a}\,[1 - 0.08331972 \cos 2\tau + 0.00114564 \cos 4\tau + 0.00007409 \cos 6\tau + 0.00000404 \cos 8\tau]\,,$$

$$r \sin v = \mathrm{a}\,[\,0.12709553 \sin 2\tau + 0.00098090 \sin 4\tau + 0.00006099 \sin 6\tau + 0.00000342 \sin 8\tau]\,.$$

$$\log \mathrm{a} = 9.5318013..$$

For $\mathrm{m} = \frac{1}{3}$;

$$r \cos v = \mathrm{a}\,[1 - 0.1622330 \cos 2\tau + 0.0048920 \cos 4\tau + 0.00059858 \cos 6\tau + 0.000081198 \cos 8\tau + 0.000011873 \cos 10\tau + 0.000001849 \cos 12\tau]\,,$$

$$r \sin v = \mathrm{a}\,[\,0.2542740 \sin 2\tau + 0.0039840 \sin 4\tau + 0.00049306 \sin 6\tau + 0.000070196 \sin 8\tau + 0.000010611 \sin 10\tau + 0.0000016902 \sin 12\tau]\,,$$

$$\log \mathrm{a} = 9.5955815.$$

For moons of much longer lunations the methods hitherto used are not practicable, and, in consequence, we resort to mechanical quadratures. Here we shall have two cases. The satellite may be started at right angles to and from a point on the line of syzygies, and the motion traced across the first quadrant; or it may be started at right angles to and from a point on the line of quadratures, and the motion traced across the second quadrant; the prime object being to discover what value of the initial velocity will make the satellite intersect perpendicularly the axis at the farther side of the quadrant.

The differential equations

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} + \left[\frac{1}{r^3} - 3\right] x = 0\,,$$

$$\frac{d^2y}{dt^2} + 2\frac{dx}{dt} + \frac{y}{r^3} = 0\,,$$

give, as expressions of the values of the coordinates, in the first case,

$$x = x_0 + 2\int_0^t y dt - \int_0^t\!\!\int_0^t \left[\frac{1}{r^3} - 3\right] x dt^2\,,$$

$$y = 2\int_0^t (x_0 - x)\, dt - \int_0^t\!\!\int_0^t \frac{y}{r^2}\, dt^2\,,$$

and, in the second case,

$$x = -2\int_0^t (y_0 - y)\,dt - \int_0^t\!\int_0^t \left[\frac{1}{r^3} - 3\right] x dt^2,$$

$$y = y_0 - 2\int_0^t x dt - \int_0^t\!\int_0^t \frac{y}{r^3}\,dt^2.$$

Here the subscript $({}_0)$ denotes values which belong to the beginning of motion, and $({}_1)$ will hereafter be used to denote those which belong to the end.

Let v be the velocity, and σ the angle, the direction of motion, relative to the rotating axes, makes with the moving line of syzygies. In the first case then $\sigma = 90°$, and we wish to ascertain what value of v_0 will make $\sigma_1 = 180°$. Generally, for small values of v_0, σ_1 will come out but little less than $270°$; but, as v_0 augments, σ_1 will be found to diminish, and, if x_0 does not exceed a certain limit, a value of v_0 can be found which will make $\sigma_1 = 180°$. In the second case, in like manner, we seek what value of v_0 will make $\sigma_1 = 270°$.

Mechanical quadratures performed with axes of coordinates having no rotation possess some advantages, as, in this case, the velocities are not present in the expressions of the second differentials of the coordinates.

Let X and Y denote the coordinates of the moon in this system, and λ its longitude measured from the line of the last syzygy, from which t is also counted. Then the potential function is

$$\Omega = \frac{1}{r} - \frac{1}{2}r^2 + \frac{3}{2}(X\cos t + Y\sin t)^2.$$

And

$$\frac{d^2X}{dt^2} = \frac{d\Omega}{dX} = -\left[\frac{1}{r^3} + 1\right] X + 3r\cos(\lambda - t)\cos t,$$

$$\frac{d^2Y}{dt^2} = \frac{d\Omega}{dY} = -\left[\frac{1}{r^3} + 1\right] Y + 3r\cos(\lambda - t)\sin t.$$

Therefore, if we compute p and θ from

$$p\cos\theta = -\left[\frac{1}{r^2} - 2r\right]\cos(\lambda - t),$$

$$p\sin\theta = -\left[\frac{1}{r^2} + r\right]\sin(\lambda - t),$$

we shall have

$$\frac{d^2X}{dt^2} = p\cos(\theta + t),$$

$$\frac{d^2Y}{dt^2} = p\sin(\theta + t).$$

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The needed values of v and σ can be derived from the equations

$$v\cos(\sigma+t)=\frac{dX}{dt}+Y,$$

$$v\sin(\sigma+t)=\frac{dY}{dt}-X.$$

The developments of the coordinates in ascending powers of t, t being counted from any desired epoch, can often be employed with advantage. Differentiating the differential equations n times we have

$$\frac{d^{n+2}x}{dt^{n+2}}=2\frac{d^{n+1}y}{dt^{n+1}}+3\frac{d^nx}{dt^n}-\frac{d^n}{dt^n}(r^{-3}x),$$

$$\frac{d^{n+2}y}{dt^{n+2}}=-2\frac{d^{n+1}x}{dt^{n+1}}-\frac{d^n}{dt^n}(r^{-3}y).$$

Also

$$\frac{d^n}{dt^n}(r^{-3}x)=r^{-3}\frac{d^nx}{dt^n}+n\frac{d(r^{-3})}{dt}\frac{d^{n-1}x}{dt^{n-1}}+\frac{n(n-1)}{1.2}\frac{d^2(r^{-3})}{dt^2}\frac{d^{n-2}x}{dt^{n-2}}+\ldots,$$

with a similar formula for the differential coefficients of $r^{-3}y$.

The differential coefficients of r^{-3}, as far as the 4th, are

$$\frac{d(r^{-3})}{dt}=-3r^{-5}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right),$$

$$\frac{d^2(r^{-3})}{dt^2}=-3r^{-5}\left(x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}+\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}\right)+15r^{-7}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)^2,$$

$$\begin{aligned}\frac{d^3(r^{-3})}{dt^3}=&-3r^{-5}\left(x\frac{d^3x}{dt^3}+y\frac{d^3y}{dt^3}+3\frac{dx}{dt}\frac{d^2x}{dt^2}+3\frac{dy}{dt}\frac{d^2y}{dt^2}\right)\\&+30r^{-7}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)\left(x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}+\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}\right)\\&-105r^{-9}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)^3,\end{aligned}$$

$$\begin{aligned}\frac{d^4(r^{-3})}{dt^4}=&-3r^{-5}\left[x\frac{d^4x}{dt^4}+y\frac{d^4y}{dt^4}+4\frac{dx}{dt}\frac{d^3x}{dt^3}+4\frac{dy}{dt}\frac{d^3y}{dt^3}+3\left(\frac{d^2x}{dt^2}\right)^2+3\left(\frac{d^2y}{dt^2}\right)^2\right]\\&+45r^{-7}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)\left(x\frac{d^3x}{dt^3}+y\frac{d^3y}{dt^3}+3\frac{dx}{dt}\frac{d^2x}{dt^2}+3\frac{dy}{dt}\frac{d^2y}{dt^2}\right)\\&+30r^{-7}\left(x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}+\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}\right)^2\\&-525r^{-9}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)^2\left(x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}+\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}\right)\\&+945r^{-11}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)^4.\end{aligned}$$

By means of these formulæ x and y can be expanded in series of ascending powers of t, as far as the term involving t^6, provided we know the values of x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ corresponding to $t=0$. Taking t sufficiently small to make the terms, involving higher powers of t than the sixth, insignificant, as, for instance, $t=0.05$ or $t=0.1$, we can ascertain the values of x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at the end of this time. With these values we can again construct new series for x and y in powers of t, in which the latter variable is counted from the end of the previous time. By repetitions of this process the integration can be carried as far as desired. Jacobi's integral, which has not been put to use in the preceding formulæ, can be employed as a check.

In case the body starts from, and at right angles to, either axis, the coefficients of every other power of t in the series for the coordinates vanish.

Thus when the axis in question is that of x, the series for the coordinates have the forms

$$x = x_0 + A_2t^2 + A_4t^4 + A_6t^6 + A_8t^8 + \ldots,$$
$$y = v_0t + A_3t^3 + A_5t^5 + A_7t^7 + A_9t^9 + \ldots.$$

By substitution of these values in the differential equations and the equating of each resulting coefficient to zero we arrive at the following equations;

$$1.2\,A_2 = 2v_0 + 3x_0 - x_0^{-2},$$
$$2.3\,A_3 = -4A_2 - x_0^{-3}v_0,$$
$$3.4\,A_4 = 6A_3 + 3A_2 + \tfrac{1}{2}x_0^{-4}(3v_0^2 + 4x_0A_2),$$
$$4.5\,A_5 = -8A_4 + \tfrac{3}{2}x_0^{-5}v_0(v_0^2 + 2x_0A_2) - x_0^{-3}A_3,$$
$$5.6\,A_6 = 10A_5 + 3A_4 + \tfrac{1}{2}x_0^{-4}(6v_0A_3 + 4x_0A_4 + 3A_2^2)$$
$$- \tfrac{3}{8}x_0^{-6}(v_0^2 + 2x_0A_2)(5v_0^2 + 6x_0A_2),$$
$$6.7\,A_7 = -12A_6 + \tfrac{3}{2}x_0^{-5}v_0(2v_0A_3 + 2x_0A_4 + A_2^2) - \tfrac{15}{8}x_0^{-7}v_0(v_0^2 + 2x_0A_2)^2$$
$$+ \tfrac{3}{2}x_0^{-5}(v_0^2 + 2x_0A_2)A_3 - x_0^{-3}A_5,$$
$$7.8\,A_8 = 14A_7 + 3A_6 + \tfrac{1}{2}x_0^{-4}(6v_0A_5 + 4x_0A_6 + 6A_2A_4 + A_3^2$$
$$- \tfrac{3}{4}x_0^{-6}(v_0^2 + 2x_0A_2)(10v_0A_3 + 8x_0A_4 + A_2^2)$$
$$+ \tfrac{5}{16}x_0^{-8}(v_0^2 + 2x_0A_2)^2(7v_0^2 + 8x_0A_2)$$
$$+ \tfrac{3}{2}x_0^{-5}(2v_0A_3 + 2x_0A_4 + A_2^2)A_2,$$

$$
\begin{aligned}
8.9\,A_9 = &- 16A_8 + \tfrac{3}{2}\,x_0^{-5}v_0\,(2v_0A_5 + 2x_0A_6 + 2A_2A_4 + A_3^2) \\
&- \tfrac{15}{4}\,x_0^{-7}v_0^2\,(v_0 + 2x_0A_2)\,(2v_0A_3 + 2x_0A_4 + A_2^2) \\
&+ \tfrac{35}{16}\,x_0^{-9}v_0\,(v_0^2 + 2x_0A_2)^3 + \tfrac{3}{2}\,x_0^{-5}\,(2v_0A_3 + 2x_0A_4 + A_2^2)\,A_3 \\
&- \tfrac{15}{8}\,x_0^{-7}\,(v_0^2 + 2x_0A_2)^2\,A_3 + \tfrac{3}{2}\,x_0^{-5}\,(v_0^2 + 2x_0A_2)\,A_5 - x_0^{-3}A_7\,.
\end{aligned}
$$

By means of these relations each A can be derived from all the A which precede it.

When the axis is that of y, the series have the forms

$$x = v_0t + A_3t^3 + A_5t^5 + A_7t^7 + A_9t^9 + \ldots\,,$$

$$y = y_0 + A_2t^2 + A_4t^4 + A_6t^6 + A_8t^8 + \ldots\,.$$

And the equations, determining the coefficients A, are

$$
\begin{aligned}
1.2\,A_2 &= -2v_0 - y_0^{-2}\,, \\
2.3\,A_3 &= 4A_2 + 3v_0 - y_0^{-3}v_0\,, \\
3.4\,A_4 &= -6A_3 + \tfrac{1}{2}\,y_0^{-4}\,(3v_0^2 + 4y_0A_2)\,, \\
4.5\,A_5 &= 8A_4 + 3A_3 + \tfrac{3}{2}\,y_0^{-5}\,v_0\,(v_0^2 + 2y_0A_2) - y_0^{-3}A_3\,.
\end{aligned}
$$

The equations are not written as far as in the former case, as it is evident they may be derived from the preceding group by putting y_0 in the place of x_0, reversing the signs of the first terms, and removing the term $3A_{n-2}$ from the equations, which give the values of the A of even subscripts, into those which give the values of the A of odd subscripts, after having augmented the subscript by unity.

The velocity of the moon of maximum lunation vanishes in quadratures, and when $v_0 = 0$ the preceding series become, putting $y_0^{-3} = \alpha$,

$$
\begin{aligned}
x = y_0 \Big[&-\frac{1}{3}\,\alpha t^3 + \left(\frac{1}{60}\,\alpha - \frac{1}{60}\,\alpha^2\right) t^5 + \left(-\frac{1}{2520}\,\alpha + \frac{1}{315}\,\alpha^2 + \frac{1}{280}\,\alpha^3\right) t^7 \\
&+ \left(\frac{1}{181440}\,\alpha - \frac{1}{12096}\,\alpha^2 + \frac{1}{45360}\,\alpha^3 + \frac{47}{9072}\,\alpha^4\right) t^9 \\
&+ \left(\frac{1}{19958400}\alpha + \frac{317}{9979200}\alpha^2 - \frac{13}{120960}\alpha^3 - \frac{10403}{4989600}\alpha^4 + \frac{947}{237600}\alpha^5\right) t^{11} \Big],
\end{aligned}
$$

$$
\begin{aligned}
y = y_0 \Big[&1 - \frac{1}{2}\,\alpha t^2 + \left(\frac{1}{6}\,\alpha - \frac{1}{12}\,\alpha^2\right) t^4 + \left(-\frac{1}{180}\,\alpha + \frac{1}{60}\,\alpha^2 - \frac{11}{360}\,\alpha^3\right) t^6 \\
&+ \left(\frac{1}{10080}\,\alpha - \frac{1}{1008}\,\alpha^2 + \frac{13}{1120}\,\alpha^3 - \frac{73}{5040}\,\alpha^4\right) t^8 \\
&+ \left(-\frac{1}{907200}\,\alpha + \frac{17}{90720}\,\alpha^2 - \frac{1}{756}\,\alpha^3 + \frac{4603}{453600}\,\alpha^4 - \frac{3}{400}\,\alpha^5\right) t^{10} \Big].
\end{aligned}
$$

These series suffice for computing the values of x and y with the desired exactitude when t is less than 0.3.

This special case of the moon of maximum lunation will now be treated. As there seems to be no ready method of getting even a roughly approximate value of y_0, we are reduced to making a series of guesses. I first took $y_0 = 0.82$; tracing the path to its intersection with the axis of x, σ_1, which ought to be 270°, came out 261° 29′ 47″.9. A second trial was made with $y_0 = 0.7937$; the result was $\sigma_1 = 267° 37' 8''.3$. Again a third trial with $y_0 = 0.7835$ gave $\sigma_1 = 269° 41' 13''.3$. The principal data acquired in the three trials are given in the following lines:

y_0.	T.	x_1.	$\frac{dx_1}{dt}$.	$\frac{dy_1}{dt}$.	σ_1.	Maximum Variation.
0.8200	0.972430	−0.339523	−0.288149	−1.927275	261° 29′ 47″.9	44° 57′ 4″
0.7937	0.908207	−0.290945	−0.089184	−2.144832	267 37 8 .3	46 39 36
0.7835	0.884782	−0.274324	−0.012170	−2.227928	269 41 13 .3	47 17 21 .

T denotes the time employed in crossing the quadrant, and the last column contains the maximum value of the angular deviation of the body from its mean direction as seen from the origin, that is, the direction it would have had, had it moved across the quadrant with a uniform angular velocity about the origin.

A check may be had on the accuracy of the computations by mechanical quadratures. We determine the value of the constant $2C$ which completes Jacobi's integral from the coordinates and velocities, both at the beginning and at the end of the motion, for each of the three trials. The result is

y_0.	First value.	Second value.
0.8200	2.34902	2.43901
0.7937	2.51985	2.51987
0.7835	2.55265	2.55261 .

We can now apply Lagrange's general interpolation formula to these data, and, regarding σ_1 as the independent variable, inquire what are the values which correspond to $\sigma_1 = 270°$. The numbers of the first trial must be multiplied by $+0.014861$; those of the second by -0.210190; those of the third by $+1.195329$, and the sums taken. The results are

y_0.	T.	x_1.	$\frac{dx_1}{dt}$.	$\frac{dy_1}{dt}$.	$2C$.	Maximum Variation.
0.781898	0.881160	0.271798	−0.000083	−2.24093	2.55788	47° 23′ 12″.

That $\frac{dx_1}{dt}$ does not rigorously vanish is due to the employment of only three terms in the interpolation; for the same reason the value of $2C$ does not quite agree with that obtained from the values of x_1 and $\frac{dy_1}{dt}$. To make all these elements accordant we add 0.00009 to the value of $\frac{dy_1}{dt}$.

A table of approximate values of x and y, derived roughly from the data afforded by the process of mechanical quadratures is appended: they will serve for plotting the orbit.

t.	x.	y.	t.	x.	y.	t.	x.	y.
0.00	—.0000	+.7819	0.30	—.0148	+.7080	0.60	—.1177	+.4748
0.02	.0000	.7816	0.32	.0180	.6978	0.62	.1294	.4519
0.04	.0000	.7806	0.34	.0215	.6869	0.64	.1418	.4277
0.06	.0001	.7790	0.36	.0256	.6752	0.66	.1547	.4022
0.08	.0003	.7767	0.38	.0301	.6629	0.68	.1680	.3752
0.10	.0005	.7737	0.40	.0351	.6499	0.70	.1818	.3466
0.12	.0009	.7701	0.42	.0407	.6361	0.72	.1956	.3162
0.14	.0015	.7659	0.44	.0468	.6216	0.74	.2095	.2839
0.16	.0022	.7610	0.46	.0534	.6063	0.76	.2230	.2496
0.18	.0032	.7554	0.48	.0607	.5902	0.78	.2359	.2131
0.20	.0044	.7492	0.50	.0686	.5733	0.80	.2475	.1745
0.22	.0058	.7432	0.52	.0771	.5555	0.82	.2575	.1339
0.24	.0076	.7347	0.54	.0863	.5369	0.84	.2653	.0913
0.26	.0096	.7265	0.56	.0961	.5172	0.86	.2704	.0474
0.28	.0120	.7176	0.58	.1066	.4965	0.88	.2718	.0027

The following is the table of the numerical values of the quantities of principal interest belonging to the moons mentioned at the beginning of this paragraph. In the first line stands the earth's moon, having very approximately $12\frac{59}{160}$ lunations in the period of its primary. In the last line is the moon of maximum lunation. The quantities belonging to the moon of two lunations have been somewhat rudely inferred from the numbers in the adjacent lines.

Number of Lunations in period of Primary. $\frac{1}{\mathrm{m}}$.	Radius Vector in Syzygies. r_0.	Radius Vector in Quadratures. r_1.	Ratio. $\frac{r_1}{r_0}$.	Velocity in Syzygies. v_0.	Velocity in Quadratures. v_1.	Ratio. $\frac{v_1}{v_0}$.	$2C$.	Maximum Variation.
$12\frac{59}{160}$	0.17610	0.17864	1.01446	2.22295	2.16484	0.97386	6.50888	0° 35′ 6″
10	0.19965	0.20418	1.02271	2.06163	1.97693	0.95892	5.88686	0 55 21
9	0.21209	0.21813	1.02849	1.98730	1.88501	0.94853	5.61562	1 9 33
8	0.22652	0.23485	1.03678	1.90904	1.78250	0.93372	5.33873	1 29 58
7	0.24342	0.25543	1.04934	1.82721	1.66572	0.91162	5.05535	2 0 53
6	0.26332	0.28167	1.06969	1.74333	1.52851	0.87677	4.76409	2 50 49
5	0.28660	0.31699	1.10605	1.66247	1.35953	0.81777	4.46103	4 18 37
4	0.31232	0.36897	1.18138	1.60111	1.13480	0.70876	4.13277	7 17 0
3	0.33235	0.45973	1.38329	1.62141	0.79387	0.48962	3.72018	14 34 14
2	0.302	0.684	2.26	2.00	0.18	0.09	2.89	37 21
1.78265	0.27180	0.78190	2.87676	2.24102	0.00000	0.00000	2.55788	47 23 12

In regard to this table we may notice the following points. The moon of the last line is the most remarkable: it is, of the class of satellites considered in this chapter, (viz., those which have the radius vector at a minimum in syzygies, and at a maximum in quadratures,) that which, having the longest lunation, is still able to appear at all angles with the sun, and thus undergo all possible phases. Whether this class of satellites is properly to be prolonged beyond this moon, can only be decided by further employment of mechanical quadratures. But it is at least certain that the orbits, if they do exist, do not intersect the line of quadratures, and that the moons describing them would make oscillations to and fro, never departing as much as 90° from the point of conjunction or of opposition.

This moon is also remarkable for becoming stationary with respect to the sun when in quadrature; and its angular motion near this point is so nearly equal to that of the sun that, for about one-third of its lunation, it is within 1° of quadrature. From the data of the table we learn that such a moon, circulating about the earth, would make a lunation in 204.896 days.

We notice that the radius vector in syzygies of this class of satellites arrives at a maximum before we reach the moon of maximum lunation. This

maximum value is very nearly, if not exactly, $\frac{1}{3}$, when measured in terms of our linear unit, and thus is a little less than double the radius vector of the earth's moon. It occurs in the case of the moon which has about 2.8 lunations in the period of its primary.

The radius vector in quadratures augments continuously as the length of the lunation increases, as also does the ratio of these radii, until, in the moon of maximum lunation, the radius in quadratures is but little less than three times that in syzygies.

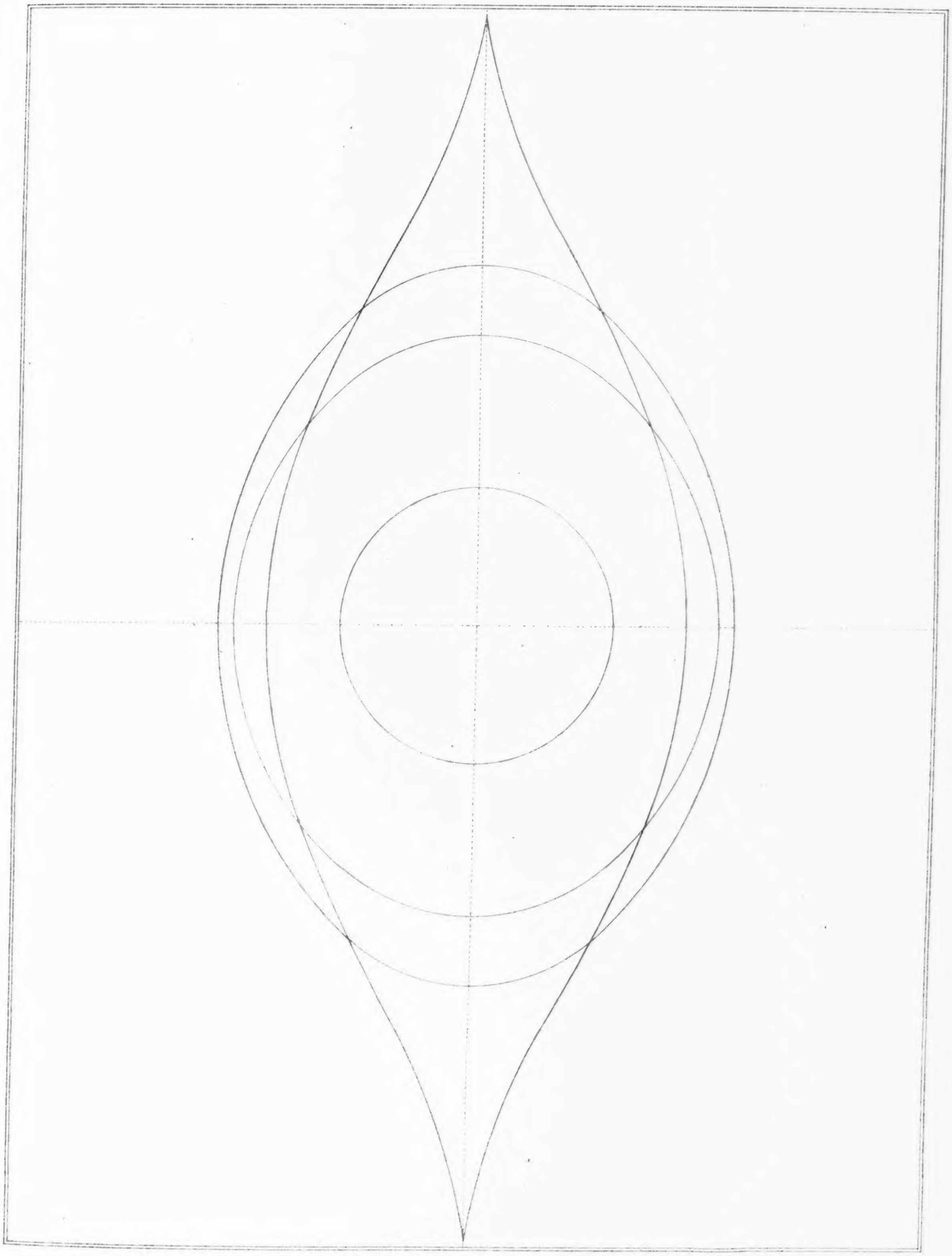
The velocity in syzygies does not continuously diminish, but attains a minimum somewhere about the moon of four lunations, and afterwards augments so that, for the moon of maximum lunation, it does not differ greatly from the velocity of the earth's moon in syzygies. On the other hand the velocity in quadratures constantly diminishes.

The maximum value of the variation augments rapidly with increase in the length of lunation, so that, in the moon of maximum lunation, it exceeds an octant, or is more than 80 times the value which belongs to the earth's moon.

In the adjoining figure are constructed graphically the paths of the earth's moon, of the moons of four and three lunations, and of the moon of maximum lunation. The moons in the first lines of the table have paths which approach the ellipse quite closely, but the paths of the moons of the last lines exhibit considerable deviation from this curve, while the orbit of the moon of maximum lunation has sharp cusps at the points of quadrature.

(To be continued.)

Plate III.



HILL, *Researches in the Lunar Theory.*

BIBLIOGRAPHY OF HYPER-SPACE AND NON-EUCLIDEAN GEOMETRY.

By George Bruce Halsted, *Tutor in Princeton College, N. J.*

Until the present century the Euclidean Geometry was supposed to be the only possible form of Space-science; that is, the space analysed in Euclid's axioms was supposed to be the only non-contradictory sort of space.

The Parallel-Postulate was generally supposed to be a consequence of the nature of straight lines, and demonstrable from the remaining postulates and axioms. The researches of Peyrard show that it was not given out by Euclid as an axiom, since in all the MSS. examined by him it is kept separate from the axioms, and has only been classed with them by an obvious error in modern times. The enormous number of unsatisfactory attempts to prove this postulate, led finally to a systematic development of the results obtainable when it is denied, and then sprang forth the Non-Euclidean or Absolute Geometry.

In the "Encyclopædie der Wissenschaften und Künste; von Ersch und Gruber; Leipzig, 1838; under "Parallel," Sohncke says that in mathematics there is nothing over which so much has been spoken, written and striven, as over the theory of parallels, and all, so far, (up to his time) without reaching a definite result and decision. He divides the attempts into three classes: — 1. In which is taken a new definition of parallels. 2. In which is taken a new axiom different from Euclid's. But just as Euclid's cannot be considered axiomatic, so is it with these new postulates. This led to the third, the largest and most desperate class of attempts, namely, to deduce the theory of parallels from reasonings about the nature of the straight line and plane angle. The article is followed by a carefully prepared list of ninety-two authors on the subject, from the earliest times up to the year 1837.

In English an account of like attempts is given in the "Geometry without Axioms," by Perronet Thompson: Cambridge, 1833; where the author also makes an elaborate attempt of his own. These accounts may be considered to bring the subject up to the point where, through the perfectly original

works of two new geometers, it assumed a totally new aspect and became the question of Non-Euclidean Geometry. At this point we take up its Bibliography, together with that of Hyper-Space, which, though springing at first from a purely analytical basis, has become intimately connected with the former.

1. LOBATCHEWSKY, NICOLAUS IVANOVITCH. (1793–1856).

The first public expression of his discoveries was given in a discourse at Kasan, February 12, 1826.

I. Principien der Geometrie. Kasan, 1829–30.

II. Neue Anfangsgruende der Geometrie, mit einer vollständigen Theorie der Parallelen. Gelehrte Schriften der Universität Kasan, 1836–38. His chief work (orig. pub. in Russian). Hoüel has made a translation of it into French (in MS.)

III. Geometrie Imaginaire. Crelle's Journal, B. XVII, pp. 295–320. 1837.

IV. Application de la Geometrie Imaginaire à quelques Integrales. Crelle. 1836.

V. Geometrische Untersuchungen zur Theorie der Parallellinien. Berlin, 1840. 61 pages.

VI. Pangeometrie, ou precis de geometrie fondée sur une theorie generale et rigoureuse des paralleles. Imprimerie de l'Université. Kazan, 1855. This, originally published in French, has been translated into Italian by G. Battaglini: Giornale di Matematiche. Anno V, Settembre e Ottobre, 1867, pp. 273–320. It is also given by Erman, Archiv Russ. XVII, 1858, pp. 397–456. V has also been translated and published in French; see Hoüel.

2. GAUSS, C. J.

I. Briefwechsel zwischen Gauss und Schumacher. See especially the letters of 17 May and 12 July, 1831. Bd. 2, pp. 268–271.

"La Géometrie non-Euclidienne ne renferme en elle rien de contradictoire, quoique, à première vue, beaucoup de ses resultats aient l'air de paradoxes. Ces contradictions apparents doivent être regardées comme l'effet d'une illusion, due à l'habitude que nous avons prise de bonne heure de considérer la géométrie Euclidienne comme rigoureuse."

II. Werke. Bd. IV, p. 215. This reference is to the researches presented in 1827 to the Society of Göttingen under the title: "Disquisitiones generales circa superficies curvas."

3. BOLYAI, WOLFGANG AND JOHANN.

I. Tentamen Juventutem studiosam in elementa Matheseos purae, elementaris ac sublimioris, methodo intuitiva, evidentique huic propria, introducendi. Tomus Primus, 1832 Secundus, 1833. 8o. Maros-Vásárhelyini. These two volumes, published by subscription, form the principal work of Wolfgang Bolyai. In the first volume, with special title page and numbering, appeared the celebrated Appendix of Johann Bolyai,

II. Ap., scientiam spatii *absolute veram* exhibens: a veritate aut falsitate Axiomatis XI Euclidei (a priori haud unquam decidenda) independentem. Auctore Johanne Bolyai de eadem, Geometrarum in Exercitu Caesareo Regio Austriaco Castrensium Captaneo. Maros-Vásárhely., 1832. (26 pages of text). This celebrated Appendix has been translated into French, see Hoüel, into Italian, see Battaglini, and into German, see Frischauf.

III. The last work of Wolfgang Bolyai, the only one he composed in German, is entitled: Kurzer Grundriss eines Versuches, I. die Arithmetik, durch zweckmässig construirte Begriffe, von eingebildeten und unendlichkleinen Grössen gereinigt, anschaulich und logisch-streng darzustellen: II. In der Geometrie, die Begriffe der geraden Linie, der Ebene, des Winkels allgemein, der winkellosen Formen, und der Krummen, der verschiedenen Arten der Gleichheit u. dgl. nicht nur scharf zu bestimmen, sondern auch ihr Sein in Raume zu beweisen: und da die Frage, *ob zwei von der dritten geschnittene Geraden, wenn die Summa der inneren Winkel nicht* $= 2R$, *sich schneiden oder nicht?*, niemand auf der Erde ohne ein Axiom (wie Euclid das XI) aufzustellen, beantworten wird; die davon unabhängige Geometrie abzusondern, und eine auf die Ja Antwort, andere auf das Nein so zu bauen, dass die Formeln der letzen auf ein Wink auch in der ersten gültig seien. Maros-Vásárhely., 1851. 8o. (88 pages of text). The author mentions Lobatchewsky's Geometrische Untersuchungen, Berlin, 1840, and compares it with the work of his son Johann Bolyai, "au sujet duquel il dit: 'Quelques exemplaires de l'ouvrage publié ici ont été envoyés à cette époque à Vienne, à Berlin, à Göttingen. . . . De Goettingen, le géant mathématique, [Gauss]

qui du sommet des hauteurs embrasse du même regard les astres et la profondeur des abîmes, a écrit qu'il était ravi de voir exécuté le travail qu'il avait commencé pour le laisser après lui dans ses papiers.' "

4. JACOBI, C. G. J.

I. De binis quibuslibet functionibus homogeneis, &c. Crelle Journ. XII, 1834. 1–69.

Several papers in the early volumes of Crelle *in effect* relate to the transformation of coordinates, and the attraction of spherical shells, &c., in n-dimensional space, but the treatment is throughout analytical and there is no especial reference to space of four or more dimensions.

5. GRASSMANN, H.

I. Die lineale Ausdehnungslehre. Leipzig. 1844. 2d Ed., 1878.

II. Die Ausdehnungslehre. Berlin, 1862.

Grassmann was perhaps the first who developed the theory of extended manifoldness, as a special case of which appears the theory of space. But his manifoldness differs from our space only as being a generalisation of it by increasing the number of dimensions while preserving relative properties of position and measure. In a word it is homaloidal Hyper-space, and does not open so wide and diverse a field as Riemann's profound paper.

6. CAYLEY, ARTHUR.

I. Chapters in the Analytical Geometry of (n) Dimensions. Camb. Math. Journ., IV, 1845. pp. 119–127.

II. Sixth Memoir upon Quantics. Phil. Trans., vol. 149.

III. On the Non-Euclidean Geometry. Clebsch, Ann. V, 630–634. 1872.

IV. A Memoir on Abstract Geometry. Phil. Trans., CLX, 51–63. 1870.

V. On the superlines of a quadric surface in five dimensional space. Quarterly Journ., vol. XII, 176–180. 1871–2.

In his Memoir on the principles of an Abstract m-dimensional Geometry (IV), Prof. Cayley says: "The science presents itself in two ways,—as a legitimate extension of the ordinary *two-* and *three-*dimensional geometries; and as a need in these geometries and in analysis generally. In fact whenever we are concerned with quantities connected together in any manner, and

which are, or are considered as variable or determinable, then the nature of the relation between the quantities is frequently rendered more intelligible by regarding them (if only two or three in number) as the coordinates of a point in a plane or in space: for more than three quantities there is, from the greater complexity of the case, the greater need of such a representation; but this can only be obtained by means of the notion of a space of the proper dimensionality; and to use such representation, we require the geometry of such space. An important instance in plane geometry has actually presented itself in the question of the determination of the number of the curves which satisfy given conditions: the conditions imply relations between the coefficients in the equation of the curve; and for the better understanding of these relations it was expedient to consider the coefficients as the coordinates of a point in a space of the proper dimensionality."

7. SYLVESTER, J. J.

I. On certain general Properties of Homogeneous Functions. Cam. and Dub. M. Journ., Feb. 1851.

II. Partitions of Numbers. (Lectures). London, 1859.

III. Barycentric Projections. Phil. Mag., or Br. Assoc'n.

IV. Inaugural Address to Math. Section, British Association at Exeter, August, 1869. Nature, vol. I, p. 238. Republished with Notes in "Laws of Verse." Longmans. 1870.

8. RIEMANN, B.

I. Ueber die Hypothesen welche der Geometrie zu Grunde liegen. Habilitationsschrift von 10 Juni, 1854. Abhandl. der König. Gesellsch. zu Göttingen. B. XIII. Reprinted in "B. Riemann's G. M. Werke," Leipzig, 1876.

This profound paper is difficult reading. Frischauf has attempted to make the study of it easier by giving, in his Absolute Geometry, notes and references at points where Riemann has given results while suppressing processes.

It has been translated into French by Hoüel. Annali di Mat., serie II, tome III, fasc. IV, 309–327. 1870.

The position of a point in space being determined by three quantities, x_1, x_2, x_3, to a continuous change of that position corresponds a continuous

variation of these three quantities. Then Riemann holds that the measure of the distance between the point $(x_1 x_2 x_3)$ and the next point $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ is not necessarily the square root of the sum of the squares of the three differentials.

If the sides of a triangle constructed on a given sphere be all of them increased or diminished in the same proportion, the shape of the triangle will not remain the same. On the contrary, the figures constructed in a plane may be magnified or diminished to any extent without alteration of shape.

Riemann found that this property of the plane is equivalent to the two following axioms: (1) That two geodesic lines which diverge from a point will never intersect again, or, as Euclid puts it, that two straight lines cannot enclose a space; and (2) that two geodesic lines which do not intersect will make equal angles with every other geodesic line. Deny the first of these axioms, and you have a manifoldness of positive curvature; deny the second, and you have one of negative curvature. The plane lies midway between the two, and its curvature is zero at every point. Thus Riemann found three different sorts of geometry. Bolyai had only noticed two. Also here for the first time was brought forward the distinction between "unbegrenzte" and "unendliche" "Unendliche" is our "infinite." A series is "unbegrenzte" when, without inversion of the derivation process, one can go on continually. If one by continued forward application of this process comes back to the starting point, the series is finite; but if the process can go on continually without ever coming again to any previous term, the series is infinite. The like parts of a circle may serve as an example of a series which though finite is yet unbegrenzt, for we may pass continually on from one to the next forever. Now we rightly attribute to space this property of being without a bound, for a limit to it is contradicted by its homogeneousness. But from this it in no way follows that space is infinite.

9. SALMON, GEORGE.

I. Lessons on Modern Higher Algebra. 1866. p. 212, &c.
II. Extension of Chasles' Theory of Characteristics to Surfaces.

10. BALTZER, R.

I. Elements of Mathematics. Dresden, 1866.
II. Ueber die Hypothesen der Parallelentheorie. C. Journal. Band 83, s. 372. Berichte der K. s. G. zu Leipzig. T. XX, 95–96. 1868.

11. HOÜEL, J.

I. Études Géométriques sur la Theorie des Parallels, par Lobatchewsky; suivi d'un extrait de la correspondance de Gauss et de Schumacher. Paris, 1866. 8o.

II. Essai critique sur les Principes fondamentaux de la Geometrie. Paris, 1867. 8o.

III. La Science Absolue de l'Espace independante de la vérité ou de la fausseté de l'Axiome XI d'Euclide (que l'on ne pourra jamais établir *a priori*); par Jean Bolyai: précédé d'une Notice sur la Vie et les Travaux de W. et J. Bolyai, par M. Fr. Schmidt. Paris, 1868. 8o.

IV. Sur les hypotheses qui servent de fondement a la geometrie, memoire posthume de B. Riemann. Annali di Mat, serie II, tome III, fasc. IV, 309–327. 1870.

V and VI. Beltrami's "Geometria non-Euclidea" and "Spazii de curvatura costante," translated into French. Annales Scien. de l'Ecole Normale Supérieure, tome VI. 1869.

VII. Note sur l'impossibilité de demontrer par une construction plane le principe de la theorie des paralleles dit Postulatum d'Euclide. Memoires de la Société des Sciences de Bordeaux, tome VIII. 1870–72. Paris: J. B. Ballière.

VIII. Du rôle de l'experience dans les sciences exactes. Prague, 1875. Translated into German by F. Müller. Grunert's Archiv., vol. 59, p. 65.

12. BELTRAMI, E.

I. Risoluzione del problema di riportare i punti di una superficie sopra un piano in modo che le linee geodetiche vengano rappresentate da linee rette. Annali di Mat., tome VII. 1866.

II. Saggio di Interpretazione della Geometria non-Euclidea. Naples, 1868. Giornale de Matematiche, (G. Battaglini,) Anno VI, pp. 284-312.

III. Teoria fondamentale degli Spazii di Curvatura costante. Annali di Mat., ser. II, tome II. Milano, 1868.

By the curvature of a system Riemann and Beltrami understand the relation of the area of an infinitesimal triangle of the system to the corresponding area of a system of constant positive curvature (système sphérique). It is in this sense that Beltrami's pseudospherical systems have a constant negative curvature.

This differs from what Kronecker calls the curvature of a system or, at the end of his Memoir, the condensation of the system, which instead of the relation of the areas of infinitesimal triangles, means the relation of the volumes of infinitesimal tetrahedrons.

IV. Theoreme de Geometrie pseudospherique. Giornale di Mat. This shows the connection between certain straight lines in the non-Euclidean plane and the curve whose tangents are of a constant length in the Euclidean plane.

V. Sur la surface de revolution qui sert de type aux surfaces pseudospheriques. Giornale di Mat. (G. Battaglini), tome X. 1872. This contains several theorems relative to the surface of revolution having for meridian the curve whose tangents are of a constant length.

13. BATTAGLINI, G.

I. Sulla Geometria Immaginaria di Lobatchewsky. Giornale di Mât., Anno V, pp. 217–231. 1867. In this the author reaches by a different method most of Lobatchewsky's results.

II. Pangeometria o sunto di geometria fondata sopra una teoria generale e rigorosa delle parallele, per N. Lobatchewsky, (versione del Francese). Giornale di Mat., Anno V, pp. 273–320.

III. Sulla scienza dello spazio assolutamente vera, ed independente dalla verita o dalla falsita dell' assioma XI di Euclide: per Giovanni Bolyai, (versione dal latino). Giornale di Mat., Anno VI, pp. 97–115. 1868.

14. HELMHOLTZ, H.

I. Ueber die Thatsachen die der Geometrie zum Grunde liegen. Nachrichten, Göttingen, Juni 3, 1868.

II. Sur les faits qui servent de base à la Geometrie. Memoires de la Soc. des Sciences de Bordeaux. 1868.

III. The Origin and meaning of Geometrical Axioms. Part I, Mind, No. III. July, 1876. Some of this article had been previously given in the Academy, Feb. 12, 1870, vol. I, p. 128. Replied to by Jevons; Nature, vol. IV, p. 481. Jevons' ideas developed by J. L. Tupper; Nature, vol. V, p. 202. Replied to by Helmholtz; Academy, vol. III, p. 52. Part II, Mind. April, 1878.

15. POTOCKI, S.

Notice historique sur la vie et les travaux de N. I. Lobatchewsky. Bulletino di Bibliographia du Prince Boncompagni, tome II, 223. May, 1869. Translated from the discourse of Janichefsky, who was editing a new edition of Lobatchewsky's works.

16. DARBOUX, G.

I. Comptes Rendus de l' Acad. Aug., 1869.

II. Sur les equations aux dérivées partielles du second ordre. Comp. R. LXX. 1870. I, 673; II, 746.

17. KRONECKER, L.

I. Ueber Systeme von Functionen mehrer Variabeln. Monatsbericht der Kgl. Akademie zu Berlin. Part I, März, 1869. Part II, August, 1869. The generalized spaces treated here are mostly supposed homaloidal. The author mentions the power given him by considerations of geometry of position in overcoming algebraical difficulties.

18. CHRISTOFFEL, E. B.

I. Allgem. Theorie d. geodät. Dreiecke. Berlin, 1869.

II. Ueber die Transformation der homogenen Differentialausdrücke 2ten Grades. Borchardt's Journal, LXX, 46–70. 1870.

III. Ueber ein betreffendes Theorem. Borchardt's Journal, LXX, 241-245. 1870. II and III treat of *n* dimensions.

19. CLIFFORD, W. K.

I. On Probability. Educational Times.

II. Lecture on "the Postulates of the Science of Space." The extent of space may be a finite number of cubic miles. He says, "In fact, I do not mind confessing that I personally have often found relief from the dreary infinities of homaloidal space in the consoling hope that, after all, this other may be the true state of things."

III. Preliminary sketch of Biquaternions. Proceedings of L. Math. Soc., IV, 381–395. The author shows that the symbols have a more general interpretation in the geometry of three dimensions which Klein calls the elliptic in distinction from the parabolic or Euclidean geometry.

20. LIPSCHITZ, R.

I. Untersuchungen in Betreff die ganzen homogenen Functionen von *n* Differentialen. Borchardt's Journal, Bde. LXX, 3, pp. 71-102. LXXII, 3, pp. 1-56. Analysed in the proceedings of the Berlin Academy, Jan., 1869, pp. 44-53. An analysis by the author is given in the Bulletin des Sciences Mathematiques, tome IV; I, pp. 97-110; II, pp. 142-157. Paris, 1873. The author demonstrates that the general form of the linear element of a system of three dimensions can be referred back to the form given by Riemann for a system of constant curvature, when a certain condition necessary and sufficient, (that the measure of the constant curvature shall be equal to a given function) is satisfied.

II. Entwickelung einiger Eigenschaften der quadratischen Formen von *n* Differentialen. Borchardt's Journal, LXXI, 274-287, 288-295. Bulletin des Sciences Math., IV, 297-307; V, 308-314. Paris, 1873.

III. Untersuchung eines Problems der Variationsrechnung. Borchardt's Journal, Bd. LXXIV, pp. 116-149, 150-171. Bulletin, tome IV, 212-224, 297-320.

IV. Extension of the Planet-problem to a space of *n* dimensions and of constant integral curvature. Translated by A. Cayley. Quar. Jour. Math. XII, 349-370. 1871.

21. GENOCCHI, A.

Dei primi principii della mecanica e della geometria in relazione al postulato d'Euclide. Firenze, 1869. Accademia da XL in Modena, serie III, tomo II, parte I. This memoir connects the theory of parallels and parallel forces with mechanical laws and considerations.

22. NÖTHER, M.

Zur Theorie der algebraischen Functionen mehrerer complexer Variabeln. Göttingen, Nachrichten. 1869.

23. BETTI, E.

Sopra gli spazi di un numero qualunque di dimensioni. Annali di Mat., 2 série, IV, pp. 140-158. 1870. Contains analytical treatment of the properties and relations of spaces of equal or different dimensions.

24. DE TILLY, M.

I. Études de mechanique abstraite. Memoires couronnés de l'Academie royale Belgique, tome XXI.

II. Report on a letter from Genocchi to Quetelet. Bulletin de Belg. (2) XXXVI, 124–139.

25. BECKER, J. K.

I. Abhandlungen aus dem Grenzgebiete der Mathematik und Philosophie. Zürich, 1870. (62 pages).

II. Ueber die neuesten Untersuchungen in Betreff unserer Anschauungen vom Raume. Schlömilch Zeitschrift, XVII, 314–332. 1872. For recension of I see XV, 93.

III. Die Elemente der Geometrie auf neuer Grundlage. Berlin, 1877. (300 pages). This contains a systematic statement of the ground-principles of the plane and of space which appear in the properties of the simplest figures. The moving idea is that all the properties of figures are grounded in the nature of space itself.

26. SCHLAEFLI, L.

I. Nota alla memoria del Sig. Beltrami sugli spazie dela curvatura costante. Annali di Mat, 2d serie, t. V, 178–193. 1870.

II. Beltrami. Osservazione sulla precedente Memoria del Sig. Prof. Schläfli. Brioschi, Ann. V, 194–198. A theorem of Beltrami leads to the problem: To distinguish all spaces of n-dimensions in which any geodetic line is represented by a system of $n - 1$ linear equations. Schläfli shows that only spaces of constant curvature fulfil this condition.

27. BEEZ, R.

I. Ueber conforme Abbildung von Mannigfaltigkeiten höherer Ordnung. Schlömilch Zeits., XX, 253–270.

II. Zur Theorie des Krummungsmasses von Mannigfaltigkeiten höherer Ordnung. Schlömilch Z., XX, 423–444. Fortsetzung, XXI, 373–401. The author shows that Kronecker's generalized expression for the measure of curvature for Hyper-space cannot, as in tridimensional space, be represented by the coefficients of the expression for the linear element. In reference to this, Lipschitz (Beitrag zur Theorie der Krummung. Borchardt's Journ.,

LXXXI, 239, Note) remarks, that to make the representation possible, one has only to take in addition "die Differentialquotienten jener Coefficienten nach den Variabeln."

28. ROSANES, J.

Ueber die neuesten Untersuchungen in Betreff unser Anschauung vom Raume. Breslau, 1871. 8o. This is an elementary exposition of the ideas contained in Riemann's celebrated paper.

29. FLYE, ST. MARIE.

I. Sur le postulatum d'Euclide. L'Institut, I, sect. XXXVIII, 53–54. 1870. That the postulatum of Euclid cannot be proved except by assuming another of like value.

II. Études analytiques sur la théorie des paralleles. Paris, 1871. 8o. Treats of a system of coordinates whose axis of x is a circle with infinite radius.

30. LIE, SOPHUS.

I. Ueber diejenige Theorie eines Raumes mit beliebig vielen Dimensionen, die der Krümmungs-Theorie des gewöhnlichen Raumes entspricht. Göttingen, Nachrichten. May, 1871. In this many geometrical theorems are extended to a space of any number of dimensions.

II. Zur Theorie eines Raumes von n Dimensionen. Göttingen, Nachrichten. Nov., 1871. 535–557. The sphere of n dimensions is used as element of a space of $n + 1$ dimensions.

31. KLEIN, FELIX.

I. Ueber die sogenannte Nicht-Euklidische Geometrie. Göttingen, Nachrichten. August, 1871. Math. Ann., IV, 573–625; VI, 112–145. The projective geometry is proved to be independent of the theorem of parallels. See Jahrbuch über die Fortschritte der Math., 1873.

II. Ueber neuere geometrische Forschungen. Erlangen, 1872.

32. SALETA, F.

Exposé sommaire de l'idée d'espace au point de vue positif. Paris, 1872. (32 pages.) The author considers the axioms and postulates of geometry as definitions of the kind of space treated.

33. KÖNIG, J.

Ueber eine reale Abbildung der Nicht-Euklidischen Geometrie. Göttingen, Nachrichten. March, 1872. (7 pages). A study of the relations which exist between the non-Euclidean geometry and the geometry of complexes.

34. JORDAN, CAMILLE.

I. Essai sur la Geometrie à *n* Dimensions. Comptes Rendus, LXXV, 1614–1617. 1872. Bulletin de la Soc. Math., tome III, pp. 104, &c., IV, p. 92.

II. Sur la theorie des courbes dans l'espace à *n* dimensions. Comptes Rendus, LXXIX, p. 795. 1874.

III. Généralisation du théorème d'Euler sur la courbure des surfaces dans l'espace à $m + k$ dimensions. Compt. Rendus, LXXIX, p. 909.

35. FRISCHAUF, J.

I. Absolute Geometrie, nach J. Bolyai. Leipzig, 1872. 8o. XII–96 pp.

II. Elemente der Absoluten Geometrie. Leipzig, 1876. VI–142 pages.

36. KOBER, J.

On infinity and the new geometry. Zeits. fur Math. Unterricht. 1872.

37. HOFFMANN, J. C. V.

Resultate der Nicht-Euklidischen oder Pangeometrie. Zeits. für Math. Unterricht, IV, 416–417.

38. FREYE, G.

Ueber ein geometrische Darstellung der imaginairen Gebilde in der Ebene. Jena. Neuenhahm.

39. CASSANI, P.

Intorno alle ipotesi fondamentali della geometria. Battaglini, G. XI, 333–349.

40. FRAHM, W.

Habilitationsschrift. Tübingen, 1873.

41. LINDEMANN, F.

Ueber unendlich kleine Bewegungen starrer Körper bei allgemeiner projectivischer Massbestimmung. Erlang., Ber., 1873, 28 Juli. Clebsch, Ann. VII, 56-144.

42. D'OVIDIO, E.

Studio sulla geometria projettiva. Brioschi, Ann. (2) VI, 72–101.

43. STAHL, H.

Ueber die Massfunctionen der analytischen Geometrie. Berlin, 1873.

44. SCHERING, E.

Linien, Flächen und höhere Gebilde im mehrfach ausgedehnten Gauss' schen und Riemann' schen Raume. Göttingen, Nachrichten, 13–21; 149–159. 1873. For a notice of the last seven by Klein, see Jahrbuch über die Fortschritte der Math. Berlin, 1875.

45. SPITZ, C.

Die ersten Sätze vom Dreiecke und den Parallelen. Nach Bolyai's Grundsätzer. Leipzig. For notice see Grunert's Archiv., LVII, Litber, CCXXVI, 10.

46. HALPHEN, G.

Recherches de géométrie à *n* dimensions. Bull. Soc. Math., F. II, 34-52. On the projective properties of structures in Hyper-space.

47. ESCHERICH.

Die Geometrie auf den Flachen constanter negativer Krummung. Kais. Akad. Bd. LXIX.

48. SPOTTISWOODE, W.

I. Sur la representation des figures de géométrie à *n* dimensions par les figures correlatives de géométrie ordinaire. Comptes Rendus, LXXI, 875–877.

II. Nouveaux exemples de la representation, par des figures de géométrie, des conceptions analytiques de géométrie à *n* dimensions. C. R., LXXXI, 961–963. Geometrical interpretation of analytical space of more than three dimensions.

49. LEWES, G. H.

Imaginary Geometry and the truth of Axioms. Problems of Life and Mind, 1st series, vol. II. London, 1875.

50. FUNCKE.

Grundlagen der Raumwissenschaft. Hannover, 1875. Proposes a case in which he holds it would be imperatively necessary to suppose a fourth dimension.

51. Zöllner, J. C. F.

I. Principien einer Elektrodynamischen Theorie der Materie. Leipzig, 1876. See Review by Prof. Carl Stumpf, Phil. Monatshefte, B. XIV, 13–30.

II. Wissenschaftliche Abhandlungen. Leipzig, 1878. See Review by P. G. Tait, Nature, March 28, 1878, pp. 420–422.

52. Frank, A. von.

Der Körperinhalt des senkrechten Cylinders und Kegels in der absoluten Geometrie. Grunert's Archiv., vol. 59, p. 76.

53. Günther, Siegmund.

Ziele und Resultate der neuern Math. Histor. Forschung. Erlangen, 1876. Also Reviews, Grunert's Archiv., Theil 60.

54. Réthy.

Die Fundamental gleichungen der nicht-Euklidischen Trigonometrie auf elementarem Wege abgeleitet. Grunert's Archiv., LVIII, 416.

55. Frankland, W.

On the simplest continuous manifoldness of two dimensions and of finite extent. Nature, vol. 15, No. 389, April 12, 1877, pp. 515–517. This article called forth an objection from C. J. Monroe in Nature, vol. 15, No. 391, April 26, 1877, p. 547, where he claims that it necessitates that a perpendicular change sign without passing through Infinity or vanishing. Of this objection Prof. Newcomb says: "I cannot see even what it consists in. The first elements of complex functions imply that a line can change direction without passing through Infinity or zero."

56. Erdmann, Benno.

Die Axiome der Geometrie. (Untersuchung der Riemann-Helmholtz Raum theorie). Leipzig, 1877.

57. Mehler.

Ueber die Benutzung einer vierfachen Manningfaltigkeit zur Ableitung orthogonaler Flachensysteme. Borchardt's Journ., Band 84, pp. 219–230. December, 1877.

58. CANTOR, G.

Ein Beitrag zur Mannigfaltigkeitslehre. Borchardt's Journ., Band 84, pp. 242–258. December, 1877.

59. NEWCOMB, SIMON.

I. Elementary theorems relating to the geometry of a space of three dimensions and of uniform positive curvature in the fourth dimension. Borchardt's Journ., Band 83, pp. 293–299. 1877.

This article, founded on the ideas of Riemann, considers the subject from the standpoint of elementary geometry. "It may also be remarked that there is nothing within our experience which will justify a denial of the possibility that the space in which we find ourselves may be curved in the manner here supposed."

II. Note on a class of Transformations which Surfaces may undergo in Space of more than three dimensions. American Journal of Math., I, pp. 1–4. 1878.

"If a fourth dimension were added to space, a closed material surface (or shell) could be turned inside out by simple flexure; without either stretching or tearing."

60. TANNERY, PAUL.

I. La Geometrie Imaginaire et la Notion d'Espace. Revue Philosophique, Nov., 1876, 433–451. II, No. 6, pp. 553-575, Juin, 1877.

61. LÜROTH, J.

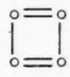
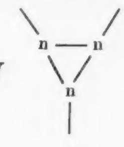
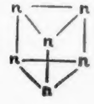
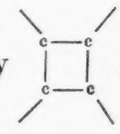
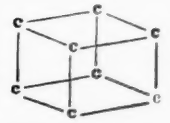
Ueber Bertrand's Beweis des Parallelenaxioms. Schlömilch Zeitschrift, XXI, pp. 294–297. 1876. Pointing out the failure of that attempted demonstration.

62. WEISSENBORN, H.

Ueber die neueren Ansichten vom Raum und von den geometrischen Axiomen. Vierteljahrschrift für Wissenschaftliche Philosophie, II, Zweites Heft, Erstes Artikel, 222–239. 1878.

SOME REMARKS ON A PASSAGE IN PROFESSOR SYLVESTER'S PAPER AS TO THE ATOMIC THEORY.

Contained in a Letter addressed to the Editors by Professor J. W. Mallet, *of the University of Virginia.*

In the paper, published in your Journal, by Professor Sylvester on "An application of the new atomic theory to the graphical representation of the invariants and covariants of binary quantics," there occurs* a suggestion which, although parenthetic only as to the main subject of the paper, seems to deserve careful consideration and more prominence than its able author has deemed it worthy of. Professor Sylvester suggests, namely, as a modification of the atomic theory of chemists in its present form, that "leaving undisturbed the univalent atoms, every other *n*-valent atom be regarded as constituted of an *n*-ad of *trivalent* atomicules arranged along the apices of a polygon of *n* sides." Thus, using small letters to stand for the proposed "atomicules," an atom of univalent hydrogen would be graphically represented by h —, and a molecule of the same element by h — h, as at present; an atom of bivalent oxygen by —o=o—, and a diatomic molecule of the same by ; an atom of trivalent nitrogen by , and a diatomic molecule by ; an atom of quadrivalent carbon by , and a diatomic molecule by ; &c.

Two important advantages of this supposition are pointed out by Professor Sylvester. First, that it furnishes a conceivable explanation of the existence, in the isolated state, of single atoms of mercury, cadmium, &c.,

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* Page 78.

which may be represented as composed of two trivalent atomicules united by all three bonds, thus

$$hg \equiv hg\,,$$

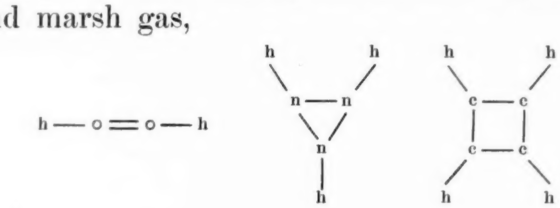
while in their ordinary compounds the same atomicules would have but two bonds *inter se*, thus

$$-\,hg = hg\,-\,.$$

Secondly. The explanation offered by Frankland of the variability of the valence of an atom (the weakest point in the theory of atomicity), viz., that "one or more pairs of bonds belonging to the atom of an element can unite and having saturated each other become, as it were, latent," has always seemed to me to carry with it absolutely no physical meaning—the reaction of a single and indivisible centre of chemical force *upon itself* may fairly be called "unthinkable." If, however, we admit Professor Sylvester's conception of an atom as made up of chemically inseparable but yet discrete atomicules, susceptible of force relation among themselves and to a variable extent, Frankland's idea assumes intelligible form.

To these results following from the suggested modification, I venture to add one or two others.

1°. A graphic representation is afforded of the difference between the old "equivalent weights" and the atomic weights of the elements; a difference which for many years formed the chief subject of controversy as to the atomic theory, and the chief stumbling block in the way of further progress in its application. Thus, if we take the old definition of the equivalent weight of an element, that it is "the smallest quantity of it which unites with 1 part of hydrogen," we have for oxygen 8, for nitrogen $4\frac{2}{3}$, for carbon 3, &c.; while other considerations, especially that of the fractional replacement of the hydrogen in such compounds as have yielded these figures, oblige us to assign the atomic weight 16 to oxygen, 14 to nitrogen, 12 to carbon, &c. If the following graphs be taken to represent the compounds in question—water, ammonia and marsh gas,



we see that the weight of the atom of oxygen being 16, of nitrogen 14, of carbon 12, the weight of the *atomicule* of oxygen is 8, of nitrogen $4\frac{2}{3}$, of carbon 3, &c.

2°. I venture to add still further a slight modification to the suggestion of Professor Sylvester.

So long as we regard atoms as the ultimate units of matter, we can, in the present state of our knowledge, assume nothing as to their size or shape;* indeed the greater number of chemists and physicists are perhaps rather inclined to look upon them as mere "centres of force." If, however, we consider an atom as made up of non-coincident atomicules, though these latter be but points, we may be quite ignorant of the size and shape of such a system of points, but both size and shape must clearly be predicable of it taken as a whole. And, in like manner (whether we adopt this idea of atomicules or not), molecules, made up of non-coincident atoms, must be possessed both of size and shape, whether these be by us determinable or not, and whether they be invariable or subject to change. Let it be assumed that an atom, when consisting of two or more atomicules, constitutes a *rigid* system, of invariable size and shape, the atomicules preserving permanently the same positions in relation to each other. But in a molecule made up of atoms, let it be assumed that the relative position of the atoms admits of change, and hence that in consequence of chemical combination, decomposition, substitution, &c., *distortion* of the molecules occurs, on which distortion may depend, in part at least, the changed properties of the masses made up of such molecules in aggregate.

If this idea be admitted I believe that an explanation may be found of the heat relations between the molecules of the two allotropic forms of oxygen, which possess special importance from their having the simplest connection with the atomic theory of all known cases of allotropism.

It is now generally agreed that the molecule of ordinary oxygen consists of two atoms, and that of ozone of three. The conversion of two molecules of the latter into three of the former is attended with evolution of heat, to an extent estimated by Berthelot from his calorimetric experiments at 59.200 heat units; and, conversely, an equivalent amount of extraneous energy must be exerted in the production of two ozone molecules from three of ordinary oxygen. If the usual graphs for these two bodies be employed, there is no reason apparent for the thermic relation in question; the number of bonds

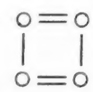
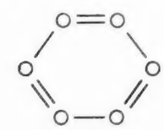
* This assertion has nothing to do with the calculations of Sir William Thomson and others as to the "size of atoms," since these calculations are only concerned with the dimensions of the *sphere of mutual interaction* of mechanical atoms or molecules, and do not at all apply to actual occupancy of space by the atoms themselves.

for each atom and the average distance between it and the other atoms with which it is connected is just the same in

 and 

(ordinary oxygen.) (ozone.)

But if Professor Sylvester's graphs be substituted, namely

 and 

(ordinary oxygen.) (ozone.)

it is at once suggested to the eye* that the average distance between the atoms, as also between the atomicules, is increased in passing from the ordinary form of oxygen to ozone; or, in other words, that the production of ozone from oxygen gas is an act of *partial* chemical decomposition, not resulting in the detachment of the constituent atoms from each other, on the contrary leaving them in a state of definite chemical combination, but removed further apart than they were in the more stable form of the element—hence, as in complete, so in this which I have called partial decomposition, extraneous energy is necessary; while the reverse change, from ozone back to ordinary oxygen, is an act of more intimate chemical union, resulting in closer approximation of the constituent parts of the molecule—and hence attended with evolution of energy, as in the form of heat.

In conclusion, I would invite the attention of mathematicians to the interesting field for their examination which is being rapidly opened up by the study of the thermic changes accompanying chemical action. The researches of Thomsen and Berthelot in particular, are fast accumulating a large mass of numerical data as to heat evolved or disappearing in connection with chemical combination and decomposition, and the highest interest attaches to a proper discussion of such results in the light of the atomic theory. In too many reviews of the facts already before us there has been gross neglect even of the thermic relations of changes of physical state accompanying chemical action, but, even where these and other collateral phe-

* It is no valid objection to this to say that the arrangement of the four atomicules in the one case in the form of a square, and of the six in the other case as a regular hexagon, is merely imaginary, for the same result as to average distance will follow from any arrangement which is *symmetrical* (and with any assumption as to relative distances between atomicules and atoms), and dealing, as we are here, solely with *similar* atoms and atomicules we can scarcely avoid the belief that their arrangement, whatever it may be, is symmetrical.

nomena have been duly allowed for, there has been so far no proper consideration of the force involved in the union or separation of *similar* as well as dissimilar atoms. Thus we find the formation of hydrochloric acid from its elements represented as

$$H + Cl = HCl = 22.000 \text{ heat units,}$$

whereas we really have

$$H_2 + Cl_2 = 2HCl = 22.000 \text{ h. u.}$$

the + 22.000 h. u. representing the algebraic sum of the thermal changes involved in the decomposition of a molecule of hydrogen (separation of its two atoms), the similar decomposition of a molecule of chlorine, and the formation of two molecules of hydrochloric acid. A mathematical discussion of the data already on hand might possibly suggest the means of so combining our calorimetric experiments as to reduce the number of unknown quantities in our equations, and lead ultimately to a clear and connected view of the force concerned in the various chemical changes which admit of being accurately examined.

UNIV. OF VIRGINIA, *July 22, 1878.*

NOTES.

I.

Historical Data concerning the Discovery of the Law of Valence.

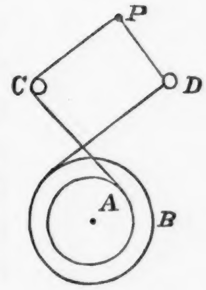
At page 89 of the Journal, reference was made to an editorial notice in *Nature* (March 14, 1875.) The date should have been February 14, and the article is of too great historical interest in connection with the Chemico-Algebraical Theory to be exposed to the chances of being overlooked. It reads as follows:

"In his interesting communication on the analogy between chemistry and algebra in our last number, Professor Sylvester attributes the conception of *valence* or *atomicity* to Kekulé. No doubt the theory in its present developed form owes much both to Kekulé and Cannizaro; indeed, until the latter chemist had placed the atomic weights of the metallic elements upon a consistent basis, the satisfactory development of the doctrine was impossible. The first conception of the theory, however, belongs to Frankland, who first announced it in his paper on Organo-metallic Bodies, read before the Royal Society on June 17th, 1852. After referring to the habits of combination of nitrogen, phosphorus, antimony and arsenic, he says: 'it is sufficiently evident, from the examples just given, that such a tendency or law prevails, and that, no matter what the character of the uniting atoms may be, *the combining power of the attracting element*, if I may be allowed the term, *is always satisfied by the same number of these atoms.*' He then proceeds to illustrate this law by the organo-compounds of arsenic, zinc, antimony, tin, and mercury. In conjunction with Kolbe, Frankland was also the first to apply this law to the organic compounds of carbon; their paper on this subject, bearing date December, 1856, having appeared in Liebig's *Annalen* in March, 1857, whilst Kekulé's first memoir, in which he mentions the tetrad functions of carbon, is dated August 15th, 1857, and was not published until November 30th in the same year. Kekulé's celebrated paper, however, in which this application of the theory of atomicity to carbon was developed, is dated March 16th, 1858, and was published on May 19th, 1858. On the other hand, the "chemicographs" or graphic formulæ, which Professor Sylvester has so successfully applied to algebra, were the invention of Crum Brown, although Frankland has used them to a much greater extent than any other chemist."

II.

On the Mechanical Description of the Cartesian.

BY J. HAMMOND, *Bath, England.*



Suppose two thin circular discs A, B rigidly attached to each other and capable of revolving round a pin through their common centre; [or perhaps it would be better to make the pin and discs revolve together.] Suppose also that fine strings, ACP, BDP, are wrapped round the discs and either passed through small rings at C, D, or twisted once round small pins there, and then knotted together at P, where a tracing point is attached.

Then it is evident that (if $CP = r$, $DP = r'$) $dr : dr' =$ ratio of the radii of A and B, and that P traces out a Cartesian, with two of its foci at C and D. When $A = B$ and one of the strings is wound on to the circle while the other is wound off, the locus of P is of course an ellipse, but when both are wound off together the locus is an hyperbola.

A simple construction for the tangent to the Cartesian at P is obtained by constructing a parallelogram of velocities, whose diagonal will be the tangent, the sides being measured along PC, PD proportional to the resolved velocities of P in those directions, *i. e.* to the radii of A and B.

III.

A New Solution of Biquadratic Equations.

BY T. S. E. DIXON, *Chicago, Ills.*

Reducing the given equation to the general form, $y^4 + 2py^2 + 8qy = r$, let $y = x + \sqrt{-\left(p + x^2 + \frac{2q}{x}\right)}$ and make the substitution. It will result in the cubic, $x^6 + px^4 + \frac{p^2 + r}{4} x^2 = q^2$, from which the values of x and consequently of y are readily obtained. Let $x = \sqrt{n}$, and the four values of y are

$$y = \sqrt{n} + \sqrt{-\left(p + n + \frac{2q}{\sqrt{n}}\right)}, \qquad y = -\sqrt{n} + \sqrt{-\left(p + n - \frac{2q}{\sqrt{n}}\right)},$$

$$y = \sqrt{n} - \sqrt{-\left(p + n + \frac{2q}{\sqrt{n}}\right)}, \qquad y = -\sqrt{n} - \sqrt{-\left(p + n - \frac{2q}{\sqrt{n}}\right)}.$$

If now we substitute for n its value derived from the cubic equation, we have a compact available formula for the expression of all the roots, free from the cumbersome $\frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$ of the old formulæ. Moreover, it is necessary to obtain only *one* value of x in the cubic equation. The equation $y^4 - 100y^2 + 480y = 576$, where n equals 9, 16 and 25, is a good illustration.

It is thus made clearly manifest that the roots of a biquadratic equation, wanting its second term, may be expressed in the general form $y = \sqrt{A} + \sqrt{B}$. This is a general expression for all four roots, since each radical has both a positive and negative root. Thus

$$y = \sqrt{A} + \sqrt{B}, \qquad y = -\sqrt{A} + \sqrt{B},$$
$$y = \sqrt{A} - \sqrt{B}, \qquad y = -\sqrt{A} - \sqrt{B}.$$

B is also a function of $\sqrt{A}$ and the expression may be written

$$y = \sqrt{A} + \sqrt{F\sqrt{A}}.$$

This leads to the suggestion that, if the roots of an equation of the sixth degree are capable of algebraic expression, they may possibly assume the form, $y = \sqrt{A} + \sqrt[3]{B}$, the two roots of the first and the three roots of the second radical affording the six necessary variations. Possibly there may be two general forms, one in which A is a function of $\sqrt[3]{B}$, and the other in which B is a function of $\sqrt{A}$.

The solution of the following equation would, however, indicate still another form. If $x^6 + px^4 + qx^3 + rx^2 + sx = \frac{s^2 - pqs + q^2r}{p^2 - 4r} = u$, then

$$x = \sqrt[3]{\frac{-q+\sqrt{q^2+4u}}{4} + \sqrt{\left(\frac{-q+\sqrt{q^2+4u}}{4}\right)^2 + \left(\frac{p}{6} + \frac{2s-pq}{6\sqrt{q^2+4u}}\right)^3}}$$
$$+ \sqrt[3]{\frac{-q+\sqrt{q^2+4u}}{4} - \sqrt{\left(\frac{-q+\sqrt{q^2+4u}}{4}\right)^2 + \left(\frac{p}{6} + \frac{2s-pq}{6\sqrt{q^2+4u}}\right)^3}}.$$

Thus, if $x^6 - 3x^4 + 60x^3 - 108x^2 - 6x = -884$, $x = -2$.

By making one or more of the coefficients p, q, r, s and u equal to zero, a number of solutions of particular cases in equations of the fifth and sixth degrees are obtained.

IV.

On a Short Process for Solving the Irreducible Case of Cardan's Method.

BY OTIS H. KENDALL, *Assistant Professor in the University of Pennsylvania.*

The equation having three commensurable roots, a, b, c, is

$$x^3 - (a + b + c)\,x^2 + (ab + ac + bc)\,x - abc = 0.$$

Reducing the roots of this equation by $\frac{1}{3}(a + b + c)$, we have

$$y^3 - \frac{1}{3}(a^2 + b^2 + c^2 - ab - ac - bc)\,y$$
$$-\frac{1}{27}(2a^3 + 2b^3 + 2c^3 - 3a^2b - 3a^2c - 3ab^2 - 3ac^2 - 3b^2c - 3bc^2 + 12abc) = 0.$$

This being of the form $y^3 + my + n = 0$, we have, substituting in Cardan's formula and reducing,

$$y = \frac{1}{3}\sqrt[3]{-\frac{27}{2}n + \frac{3}{2}(a-b)(a-c)(b-c)\sqrt{-3}}$$
$$+\frac{1}{3}\sqrt[3]{-\frac{27}{2}n - \frac{3}{2}(a-b)(a-c)(b-c)\sqrt{-3}} = u + v.$$

Since u is a binomial imaginary, its cube root will be of the form $\alpha + \sqrt{-\beta}$, and $\sqrt{\frac{\beta}{3}}$ will be rational. Hence

$$(1) \qquad \alpha(\alpha^2 - 3\beta) = \frac{1}{2}(2a^3 + 2b^3 + 2c^3 - 3a^2b - 3a^2c - 3ab^2 - 3ac^2 - 3b^2c - 3bc^2 + 12abc),$$

$$(2) \qquad \sqrt{-\beta}\,(3\alpha^2 - \beta) = \frac{3}{2}(a-b)(a-c)(b-c)\sqrt{-3} = (r).$$

Since α must be rational, $\sqrt{-\beta}$ must be of the first degree with reference to a, b, c, and the only factors of (r) of that degree are of the form $\frac{(a-b)\sqrt{-3}}{p}$, where p is some integer: p must be 2, for substituting $\sqrt{-\beta} = \frac{b-c}{p}\sqrt{-3}$ in (2) and reducing, we have

$$\alpha^2 = \frac{p}{2}(a^2 - ab - ac + bc) + \frac{b^2 - 2bc + c^2}{p^2},$$

which will not give a rational value to α unless $p = 2$.

Let us assume then $\sqrt{-\beta} = \frac{b-c}{2}\sqrt{-3}$, substitute in (2) and reduce. We find $\alpha = \pm\left(a - \frac{b+c}{2}\right)$, and by substitution in (1), $\alpha = a - \frac{b+c}{2}$, and similarly for the other factors of (r). Hence

$u = \frac{1}{3}\left(a - \frac{b+c}{2} + \frac{b-c}{2}\sqrt{-3}\right),\ \frac{1}{3}\left(b - \frac{a+c}{2} + \frac{a-c}{2}\sqrt{-3}\right)$, and $\frac{1}{3}\left(c - \frac{a+b}{2} + \frac{a-b}{2}\sqrt{-3}\right)$; similarly

$v = \frac{1}{3}\left(a - \frac{b+c}{2} - \frac{b-c}{2}\sqrt{-3}\right),\ \frac{1}{3}\left(b - \frac{a+c}{2} - \frac{a-c}{2}\sqrt{-3}\right)$, and $\frac{1}{3}\left(c - \frac{a+b}{2} - \frac{a-b}{2}\sqrt{-3}\right)$;

$y = \frac{1}{3}(2a - b - c),\ \frac{1}{3}(2b - a - c)$, and $\frac{1}{3}(2c - a - b)$; $x = a, b$, and c. The three values of u are connected, as they should be, by the ratios

$$1 : \frac{1}{2}(-1 + \sqrt{-3}) : \frac{1}{2}(-1 - \sqrt{-3}).$$

If two of the roots of the equation are equal, as b and c, then $(r) = 0$ and $-\frac{1}{2}n = (a - b)^3$; and if two are imaginary, as b and c, $(= \gamma \pm \delta\sqrt{-3})$, (r) becomes rational and $-\frac{1}{2}n + (r) = (a - \gamma - 3\delta)^3$, showing why, in these two cases, Cardan's formula gives a rational result.

Numerical Equations.

1. $x^3 - 9x^2 + 14x + 24 = 0$.

Reduce the roots by 3; then $y^3 - 13y + 12 = 0$, and by Cardan's formula,

$$y = \sqrt[3]{-6 + \frac{35}{3}\sqrt{-\frac{1}{3}}} + \sqrt[3]{-6 - \frac{35}{3}\sqrt{-\frac{1}{3}}};$$

(1) $\alpha(\alpha^2 - 3\beta) = -6$, (2) $\sqrt{-\beta}\,(3\alpha^2 - \beta) = \frac{35}{3}\sqrt{-\frac{1}{3}}$;

$\sqrt{-\beta} = \sqrt{-\frac{1}{3}},\ \frac{5}{2}\sqrt{-\frac{1}{3}}$, and $-\frac{7}{2}\sqrt{-\frac{1}{3}}$; $\beta = \frac{1}{3},\ \frac{25}{12}$, and $\frac{49}{12}$; then by (2), $\alpha = \pm 2,\ \pm\frac{3}{2}$, and $\pm\frac{1}{2}$; and by (1), $\alpha = -2,\ \frac{3}{2}$, and $\frac{1}{2}$.

$\therefore\ y = -2 + \sqrt{-\frac{1}{3}} + \left(-2 - \sqrt{-\frac{1}{3}}\right) = -4$, $y = \frac{3}{2} + \frac{5}{2}\sqrt{-\frac{1}{3}} + \frac{3}{2} - \frac{5}{2}\sqrt{-\frac{1}{3}} = 3$, and $y = \frac{1}{2} - \frac{7}{2}\sqrt{-\frac{1}{3}} + \frac{1}{2} + \frac{7}{2}\sqrt{-\frac{1}{3}} = 1$; $x = -1$, 6, and 4.

2. $x^3 - 16x^2 + 73x - 90 = 0$.

Reduce the roots by $\frac{16}{3}$; then $y^3 - \frac{37}{3}y - \frac{110}{27} = 0$, and by Cardan's formula,

$$y = \frac{1}{3}\sqrt[3]{55 + 126\sqrt{-3}} + \frac{1}{3}\sqrt[3]{55 - 126\sqrt{-3}}.$$

Since $\frac{126}{27}\sqrt{-3} = \frac{3}{2}(a-b)(b-c)(a-c)\sqrt{-3}$, the factors are probably $\pm 2\sqrt{-3}$, $\pm\frac{3}{2}\sqrt{-3}$ and $\pm\frac{7}{2}\sqrt{-3}$. Hence

(1) $\quad \alpha(\alpha^2 - 3\beta) = 55, \qquad$ (2) $\quad \sqrt{-\beta}\,(3\alpha^2 - \beta) = 126\sqrt{-3};$

$$\sqrt{-\beta} = 2\sqrt{-3}; \quad 3\alpha^2 = 75, \quad \alpha = \pm 5; \quad \alpha = -5;$$

$$\sqrt{-\beta} = \frac{3}{2}\sqrt{-3}; \quad 3\alpha^2 = \frac{363}{4}, \quad \alpha = \pm\frac{11}{2}; \quad \alpha = \frac{11}{2};$$

$$\sqrt{-\beta} = -\frac{7}{2}\sqrt{-3}; \quad 3\alpha^2 = \frac{3}{4}, \quad \alpha = \pm\frac{1}{2}; \quad \alpha = -\frac{1}{2};$$

$$y = \frac{1}{3}(-5 + 2\sqrt{-3}) + \frac{1}{3}(-5 - 2\sqrt{-3}) = -\frac{10}{3},$$

$$y = \frac{1}{3}\left(\frac{11}{2} + \frac{3}{2}\sqrt{-3}\right) + \frac{1}{3}\left(\frac{11}{2} - \frac{3}{2}\sqrt{-3}\right) = \frac{11}{3},$$

$$y = \frac{1}{3}\left(-\frac{1}{2} - \frac{7}{2}\sqrt{-3}\right) + \frac{1}{3}\left(-\frac{1}{2} + \frac{7}{2}\sqrt{-3}\right) = -\frac{1}{3};$$

$$x = 2,\ 9, \text{ and } 5.$$

V.

An Extension of Taylor's Theorem.

By J. C. Glashan, *Ottawa, Canada.*

$$f(x+a) - f(x) = \int_0^a da\,\frac{d}{dx}f(x+a), \quad \left(1 - \int_0^a da\,\frac{d}{dx}\right)f(x+a) = f(x),$$

$$\therefore f(x+a) = \left(1 - \int_0^a da\,\frac{d}{dx}\right)^{-1} f(x). \qquad \text{I.}$$

$$\left(1 - \int_0^a da\,\frac{d}{dx}\right)^{-1}\int_0^a da\,f(x+b) = \int_0^a da\left(1 - \int_0^a da\,\frac{d}{dx}\right)^{-1} f(x+b)$$

(by I,)
$$= \int_0^a da\left\{1 - \int_0^{a+b} d(a+b)\,\frac{d}{dx}\right\}^{-1} f(x). \qquad \text{II.}$$

Expanding by I, but modifying each remainder by II before proceeding to obtain the term next following, we get

$$f(x+a+b+c+e+\&c.) = f(x+b+c+e+\&c.) + \int_0^a da\,f'(x+c+e+\&c.)$$
$$+ \int_0^a da\int_0^{a+b} d(a+b)\,f''(x+e+\&c.)$$
$$+ \int_0^a da\int_0^{a+b} d(a+b)\int_0^{a+b+c} d(a+b+c)\,f'''(x+\&c.) + \&c., \qquad \text{III.}$$

which is the extension proposed.

Writing ${}^{m}C$ for a function of $a_0, a_1, a_2, \ldots a_{m-1}$, and ${}^{n-m}\omega\,.\,{}^{m}C$ for the result of substituting, in ${}^{m}C$, $a_1 + a_2 + a_3 + \ldots a_{n-m}$ for a_0, a_{n-m+1} for a_1, a_{n-m+2} for a_2, &c., that is, if ${}^{m}C \equiv F(a_0, a_1, a_2, \ldots . a_{m-1})$, then

$${}^{n-m}\omega\,.\,{}^{m}C \equiv F(a_1 + a_2 + a_3 + \ldots a_{n-m}, a_{n-m+1}, a_{n-m+2}, \ldots a_{n-1}),$$

and determining these C-functions by

$${}^{1}C \equiv a_0,$$

$${}^{n}C \equiv a_0^n + \frac{n}{1} a_0^{n-1} ({}^{n-1}\omega\,.\,{}^{1}C) + \frac{n(n-1)}{1\,.\,2} a^{n-2} ({}^{n-2}\omega\,.\,{}^{2}C) + \ldots . + \frac{n}{1} a_0 ({}^{1}\omega\,.\,{}^{n-1}C),$$

III may be put under the form

$$f(x) = f(x - a_0) + \frac{{}^{1}C}{1} f'(x - a_0 - a_1) + \frac{{}^{2}C}{1\,.\,2} f''(x - a_0 - a_1 - a_2) + \ldots .. $$

$$\ldots .. + \frac{{}^{n}C}{n!} f^n (x - \Sigma_0^n a_r) + \&c. \qquad \text{IV.}$$

Writing a for a_0, then if $a_1 = a_2 = a_3 = \ldots = a_n = b$, this becomes the Series of Abel given by M. J. Bertrand in his *Traité de Calcul differentiel*, p. 324.

$$f(x) = f(x - a) + \frac{a}{1} f(x - a - b) + \frac{a(a + 2b)}{1\,.\,2} f\{x - (a + 2b)\} + \ldots .$$

$$\ldots .. + \frac{a(a + nb)^{n-1}}{n!} f\{x - (a + nb)\} + \&c.$$

If $b = 0$ this becomes Taylor's Theorem, or the simple expansion of I as evidently it should.

Using a, b, c, d, e, &c., instead of a_0, a_1, a_2, &c., we get for the first five coefficients of IV,

$${}^{1}C \equiv a,$$

$${}^{2}C \equiv a^2 + 2ab,$$

$${}^{3}C \equiv a^3 + 3a^2(b + c) + 3a(b^2 + 2bc),$$

$${}^{4}C \equiv a^4 + 4a^3(b + c + d) + 6a^2\{(b + c)^2 + 2(b + c)d\} + 4a\{b^3 + 3b^2(c + d) + 3b(c^2 + 2cd)\},$$

$${}^{5}C \equiv a^5 + 5a^4(b + c + d + e) + 10a^3\{(b + c + d)^2 + 2(b + c + d)e\} + 10a^2\{(b + c)^3 + 3(b + c)^2(d + e) + 3(b + c)(d^2 + 2de)\} + 5a[b^4 + 4b^3(c + d + e) + 6b^2\{(c + d)^2 + 2(c + d)e\} + 4b\{c^3 + 3c^2(d + e) + 3c(d^2 + 2de)\}].$$

THÉORIE DES FONCTIONS NUMÉRIQUES SIMPLEMENT PÉRIODIQUES.

PAR EDOUARD LUCAS, *Professeur au Lycée Charlemagne, Paris.*

(Voir pag. 240 et suiv.)

SECTION XXIV.

De l'apparition des nombres premiers dans les séries récurrentes de première espèce.

Dans les séries récurrentes de première espèce, a et b désignent deux nombres entiers, positifs et premiers entre eux; il est d'abord évident que les diviseurs premiers de a et de b, ou de $Q = ab$, ne se trouvent jamais comme facteurs dans la série; il ne sera pas tenu compte de ces diviseurs dans tout ce qui va suivre. On déduit immédiatement de la première des formules (4), la démonstration du théorème de FERMAT. En effet, on a, en négligeant les multiples de p, supposé premier et impair,

$$2^{p-1}\frac{a^p - b^p}{a - b} \equiv \delta^{p-1}, \text{ (Mod. } p).$$

Multiplions les deux termes de la congruence par $\delta = a - b$, nous obtenons

$$2^{p-1}(a^p - b^p) \equiv (a - b)^p, \text{ (Mod. } p);$$

supposons $a - b = 2$, et divisons par 2^{p-1}, il vient

$$a^p - b^p \equiv a - b, \text{ (Mod. } p),$$

ou, encore

$$a^p - a \equiv b^p - b, \text{ (Mod. } p).$$

Ainsi, le reste de la division de $a^p - a$, par p premier, ne change pas lorsque l'on diminue a de deux unités, et par suite de 2, 4, 6, 8,... unités; mais pour $a = 0$ ou $a = 1$, ce reste est nul; donc $a^p - a$ est toujours divisible par le nombre premier p, quelque soit l'entier a. Par suite, si le nombre entier a n'est pas divisible par p, la différence $a^{p-1} - 1$ est divisible par p; c'est précisément l'énoncé du théorème en question.

En supposant maintenant a et b quelconques, mais non divisibles par p, les différences

$$a^{p-1} - 1 \text{ et } b^{p-1} - 1$$

sont divisibles par p; donc, si $a - b$ n'est pas divisible par p, on a

$$U_{p-1} = \frac{a^{p-1} - b^{p-1}}{a - b} \equiv 0, \text{ (Mod. } p).$$

Par conséquent, les différents termes des séries récurrentes de première espèce contiennent, en exceptant les diviseurs de $Q = ab$ et de $\delta = a - b$, tous les nombres premiers en facteurs.

Mais, s'il est vrai que p divise U_{p-1}, on peut, dans la plupart des cas, trouver un terme de rang inférieur à $p - 1$, et divisible par p. Désignons par ω le *rang d'arrivée* ou *d'apparition* du nombre premier p dans la série des U_n; il résulte des principes exposés (Section XI), que l'on a, pour k entier et positif,

$$U_{k\omega} \equiv 0, \text{ (Mod. } p);$$

ainsi, tous les termes divisibles par p ont un rang égal à un multiple quelconque du rang d'apparition.

Il résulte encore des principes exposés (Section XIII), que les termes, dont le rang est un multiple quelconque de $(p-1)\,p^{\lambda-1}$, sont divisibles par p^λ; mais il peut exister d'autres termes divisibles par p^λ, pour deux raisons bien différentes; 1° lorsque le rang d'arrivée ω de p diffère de $p - 1$; 2° lorsque le nombre premier p arrive pour la première fois à une puissance supérieure à la première; mais, cela connu, il est facile de tenir compte de ces singularités. En général, si m désigne un nombre quelconque premier avec Q, et $\varphi(m)$ *l'indicateur de* m, c'est-à-dire le nombre des entiers inférieurs et premiers à m, on a la congruence

$$U_{\varphi(m)} \equiv 0, \quad \text{(Mod. } m); \tag{135}$$

cette congruence correspond au *théorème de* FERMAT *généralisé* par EULER. Inversement, si l'on a

$$U_n \equiv 0, \quad \text{(Mod. } m),$$

on en déduit

$$n = k\mu,$$

μ désignant un certain diviseur de $\varphi(m)$, et k un entier positif quelconque.

Les résultats que nous venons d'obtenir conduisent à la forme linéaire des diviseurs premiers de U_n. En effet, si ω désigne toujours le rang d'arrivée de p, on a, puisque U_{p-1} est divisible par p,

$$p - 1 = k_0\omega,$$

et, par suite

$$p = k_0\omega + 1. \tag{136}$$

Nous appellerons *diviseurs propres* de U_n tous les facteurs premiers de U_n que l'on ne rencontre pas dans les termes de rang inférieur, et *diviseurs impropres,* les facteurs premiers contenus préalablement dans les termes de la série. On a alors les deux propositions suivantes:

THÉORÈME I: *Les diviseurs impropres des termes U_n des fonctions simplement périodiques sont des diviseurs propres des termes dont le rang est un diviseur de n.*

THÉORÈME II: *Les diviseurs propres des termes U_n des fonctions périodiques de première espèce appartiennent à la forme linéaire $kn+1$.*

Enfin, si l'on observe que l'on a trouvé

$$U_{2n} = U_n V_n,$$

on a encore:

THÉORÈME III: *Les diviseurs propres de V_n appartiennent à la forme linéaire* $2kn+1$.

On déduit encore de ce qui précède la démonstration du théorème suivant, qui n'est qu'un cas particulier du théorème de LEJEUNE-DIRICHLET, sur les progressions arithmétiques:

THÉORÈME IV: *Quel que soit l'entier m, il y a une série indéfinie de nombres premiers de la forme linéaire $km+1$.*

En effet, il est d'abord évident que, pour une valeur suffisamment grande de n, le terme U_n possède nécessairement un ou plusieurs diviseurs propres de la forme $kn+1$. Par conséquent, si l'on fait successivement n égal à

$$m,\ pm,\ p^2m,\ p^3m,\ \ldots\ p^\lambda m,$$

p étant premier, les termes correspondants possèdent tous, à partir d'un certain rang, des diviseurs de la forme considérée; le théorème est donc démontré.

Il résulte encore, du théorème I (Section XX), que ces diviseurs appartiennent en outre aux diviseurs de la forme quadratique $x^2 \pm py^2$, suivant que l'on prend pour p un nombre premier de la forme $4q+3$, ou de la forme $4q+1$.

Les théorèmes précédents permettent encore de déterminer les diviseurs des fonctions numériques de première espèce; nous donnerons d'abord les deux exemples suivants dus à EULER.

EXEMPLE I: Soit, dans la série de FERMAT,

$$U_{64} = 2^{64} - 1 = 18446\ 74407\ 37095\ 51615,$$

on a, d'après les formules précédentes,

$$U_{64} = U_1 V_1 V_2 V_4 V_8 V_{16} V_{32};$$

on a immédiatement les décompositions en facteurs premiers

$$U_1 = 1, \quad V_1 = 3, \quad V_2 = 5, \quad V_4 = 17, \quad V_8 = 257, \quad V_{16} = 65537;$$

et

$$V_{32} = 42949\,67297.$$

Les diviseurs de V_{32} appartiennent à la forme linéaire $64k + 1$; en essayant les diviseurs premiers de cette forme

$$193, \ 257, \ 449, \ 577, \ 641,$$

on trouve

$$V_{32} = 641 \times 67\,00417.\,*$$

L'essai des diviseurs premiers de même forme

$$641, \ 769, \ 1153, \ 1217, \ 1409, \ 1601, \ 2113,$$

et inférieurs à la racine carrée du second facteur de V_{32}, indique presque immédiatement que 67 00417 est un nombre premier.

FERMAT avait annoncé, mais sans dire qu'il en eût la démonstration, dans une lettre du 18 Octobre 1640, que la formule $2^{2^n} + 1$ donnait toujours des nombres premiers. Cette formule se trouve en défaut, d'après la décomposition précédente, due à EULER, pour $n = 5$.

On sait, d'autre part, que GAUSS a démontré que l'on peut diviser la circonférence en $2^{2^n} + 1$ parties égales, lorsque ce nombre est premier, et seulement dans ce cas, par la règle et le compas. Nous indiquerons plus loin une méthode de recherche du mode de composition des nombres de cette forme, basée sur la distribution des nombres premiers dans la série de PELL. Par la méthode que nous venons d'exposer, en supposant que le nombre

$$2^{2^6} + 1 = 18446\,74407\,37095\,51617,$$

soit premier, il faudrait à un seul calculateur, pour le démontrer, tout en profitant de la forme $128k + 1$, imposée aux diviseurs de ce nombre, environ *trois mille ans* de travail assidu.† Par notre méthode, il suffit de *trente heures*, pour décider si ce nombre est premier ou composé.

EXEMPLE II: Soit encore, dans la série de FERMAT, le terme

$$U_{31} = 2^{31} - 1 = 21474\,83647$$

dont le rang 31 est un nombre premier. Les diviseurs de U_{31} sont, sans exception, des diviseurs propres appartenant à la forme linéaire $62k + 1$. Mais, d'autre part (Section VIII, Théorème I), en tenant compte des formes quadratiques de ses diviseurs, ou des formes linéaires correspondantes $8k' \pm 1$,

* Il est inutile, d'après la loi de répétition, d'essayer 257 qui se trouve dans V_8. Nous avons démontré que les diviseurs de V_{32} appartiennent à la forme $128k + 1$. (*Académie de Turin*, janvier 1878)

† *Aux mathématiciens de toutes les parties du monde.—Communication sur la décomposition des nombres en leurs facteurs simples.* Par M. F. LANDRY. Paris, 1867. (Note de la page 8.)

on voit que tout diviseur premier de U_{31} appartient nécessairement à l'une des formes linéaires

$$248k + 1, \quad 248k + 63.$$

"Or, EULER* nous apprend qu'après avoir essayé tous les nombres premiers contenus dans ces deux formes, jusqu'à 46339, racine du nombre $2^{31}-1$, il n'en a trouvé aucun qui fût diviseur de ce nombre; d'où il faut conclure conformément à une assertion de FERMAT, que le nombre $2^{31}-1$ est un nombre premier. C'est le plus grand de ceux qui aient été vérifiés jusqu'à présent." (LEGENDRE, Théorie des Nombres, 3^e^ édition, t. I, pag. 229. Paris, 1830.)

EXEMPLE III: On connaissait, depuis quelques années, un nombre premier plus grand que le précédent, indiqué par PLANA, dans son *Mémoire sur la Théorie des Nombres*, du 20 Novembre 1859.† Soit, en effet

$$V_{29} = 3^{29} + 1;$$

ce nombre a tous ses diviseurs propres de la forme $58k + 1$; mais d'autre part, ces diviseurs appartiennent à la forme quadratique $x^2 + 3y^2$, et, par suite, aux formes linéaires $12k + 1$ et $12k + 7$. En combinant l'une de ces formes avec la précédente, on trouve que les diviseurs de V_{29} sont de l'une des deux formes

$$348k + 1, \quad \text{ou} \quad 348k + 175.$$

PLANA a ainsi trouvé la décomposition

$$V_{29} = 2^2 \times 6091 \times 28168\,76431,$$

et vérifié que le dernier facteur est premier. Il a encore indiqué (*loc. cit.*, pag. 140 et 141) que le quotient

$$\frac{3^{29}-1}{2 \times 59} = 58\,16133\,67499,$$

n'a pas de diviseur premier inférieur à 52259, et que le nombre $2^{53}-1$ n'a pas de diviseur inférieur à 50033. Ces trois assertions sont inexactes; on a

$$3^{29} - 1 = 2 \times 59 \times 28537 \times 203\,81027,$$
$$3^{29} + 1 = 2^2 \times 523 \times 6091 \times 53\,85997,$$
$$2^{53} - 1 = 6361 \times 69431 \times 203\,94401.$$

Nous ajouterons que l'on trouve encore dans la mémoire de PLANA, la décomposition

$$2^{41} - 1 = 13367 \times 1645\,11353.$$

* *Lettre à Bernoulli*, en 1771,—*Mémoires de l'Académie de Berlin*, année 1772, pag. 36.

†*Memorie della Reale Accademia delle Scienze di Torino*, 2^e^ série, t. XX, p. 139. Turin, 1863.

EXEMPLE IV: Nous donnerons encore quelques exemples de décomposition de la fonction numérique

$$(2m)^{2m} - 1,$$

qui joue un rôle assez important dans les congruences de degré supérieur. Nous avons trouvé les résultats suivants:

$$\left\{\begin{array}{l}
14^7 - 1 = 13 \times 81\,08731, \\
14^7 + 1 = 3 \times 5 \times 70\,27567, \\
20^{10} - 1 = 3 \times 7 \times 11 \times 19 \times 61 \times 251 \times 1\,52381, \\
20^{10} + 1 = 41 \times 401 \times 2801 \times 2\,22361, \\
22^{11} - 1 = 3 \times 7 \times 67 \times 353 \times 11764\,69537, \\
22^{11} + 1 = 23 \times 89 \times 28\,54510\,51007, \\
24^{12} - 1 = 5^2 \times 7 \times 13 \times 23 \times 73 \times 79 \times 349 \times 577 \times 601, \\
24^{12} + 1 = 97 \times 3\,31777 \times 11347\,93633, \\
28^{14} - 1 = 3^3 \times 29 \times 113 \times 13007 \times 35771 \times 44\,22461, \\
30^{15} - 1 = 7^2 \times 19 \times 29 \times 12211 \times 8\,37931 \times 519\,41161, \\
30^{15} + 1 = 11 \times 13 \times 31 \times 67 \times 271 \times 4831 \times 71261 \times 5\,17831,
\end{array}\right.$$

dont nous donnerons plus tard l'application à de nouvelles recherches sur le dernier théorème de FERMAT.

SECTION XXV.

De l'apparition des nombres premiers dans les séries récurrentes de seconde et de troisième espèce.

En désignant toujours par p un nombre premier quelconque, on sait que le reste de la division de $\Delta^{\frac{p-1}{2}}$ par p est toujours égal à 0, à $+1$, ou à -1, suivant que Δ est un multiple, un résidu quadratique, ou un non-résidu quadratique de p. Nous considérerons les cinq cas suivants.

PREMIER CAS. *p est un diviseur de P.*

On a $U_2 = P$, et par conséquent tous les termes U_n de rang pair de la série sont divisibles par p; en désignant par p^λ la plus haute puissance de p qui divise P, les rangs des termes divisibles par $p^{\lambda+\mu}$ seront tous les multiples de $2p^\mu$.

DEUXIÈME CAS. *p est un diviseur de Q.*

Nous avons, par définition,

$$2^n \sqrt{\Delta}\, U_n = (P + \sqrt{\Delta})^n - (P - \sqrt{\Delta})^n,$$
$$2^n V_n = (P + \sqrt{\Delta})^n + (P - \sqrt{\Delta})^n;$$

on a donc, en supprimant les multiples de Q, par le remplacement de Δ par P^2, les congruences

$$2^n P U_n \equiv (P + P)^n, \quad (\text{Mod. } Q),$$
$$2^n V_n \equiv (P + P)^n, \quad (\text{Mod. } Q),$$

ou, plus simplement,

(137) $$U_n \equiv P^{n-1}, \quad V_n \equiv P^n, \quad (\text{Mod. } Q).$$

Par conséquent, U_n et V_n ne sont jamais divisibles par Q ou par l'un de ses diviseurs, puisque P et Q ont été supposés premiers entre eux. D'ailleurs ce résultat s'applique aux séries de première et de troisième espèce ; lorsque l'on a $Q = \pm 1$, comme dans les séries de PELL et de FIBONACCI, nous n'aurons pas à tenir compte du théorème précédent.

TROISIÈME CAS. *p est un diviseur de Δ.*

Lorsque p est un nombre premier diviseur de Δ, les formules (4) donnent immédiatement,

(138) $$U_p \equiv 0, \quad V_p \equiv P, \quad (\text{Mod. } p).$$

et, par suite cette proposition :

THÉORÈME : *Dans la série U de seconde espèce, tout diviseur premier p du déterminant Δ est un diviseur de U_p.*

Il résulte d'ailleurs des principes exposés précédemment, qu'un diviseur premier impair p de Δ arrive pour la première fois, dans U_p et *à la première puissance.*

QUATRIÈME CAS. *Δ est résidu quadratique de p.*

En changeant, dans la première des formules (4), p en $p - 1$, on a

$$2^{p-2} U_{p-1} = \frac{p-1}{1} P^{p-2} + \frac{(p-1)(p-2)(p-3)}{1.2.3} P^{p-4}\Delta + \ldots. + \frac{p-1}{1} P\Delta^{\frac{p-3}{2}};$$

et, en appliquant les résultats obtenus (Section XXI) pour les congruences du triangle arithmétique, on a

$$2^{p-2} U_{p-1} \equiv -\left[P^{p-2} + P^{p-4}\Delta + P^{p-6}\Delta^2 + \ldots + P\Delta^{\frac{p-3}{2}}\right], \quad (\text{Mod. } p),$$

et, par suite

$$2^{p-2} U_{p-1} \equiv - P \frac{P^{p-1} - \Delta^{\frac{p-1}{2}}}{P^2 - \Delta}, \quad (\text{Mod. } p).$$

Mais on a, par le théorème de FERMAT, $P^{p-1} \equiv 1$, (Mod. p), et, puisque Δ est résidu quadratique de p, il en résulte que U_{p-1} est divisible par p. On a donc cette proposition, qui s'applique aux séries de troisième espèce, en tenant compte du signe de Δ:

THÉORÈME : *Dans la série récurrente U de seconde ou de troisième espèce, tout nombre premier p, qui admet Δ pour résidu quadratique, divise le terme U_{p-1}.*

La seconde des formules (4) donne

$$2^{p-2} V_{p-1} = P^{p-1} + \frac{(p-1)(p-2)}{1.2} P^{p-3}\Delta + \ldots + \Delta^{\frac{p-1}{2}},$$

et, par suite

$$2^{p-2} V_{p-1} \equiv P^{p-1} + P^{p-3}\Delta + \ldots + \Delta^{\frac{p-1}{2}}, \quad (\text{Mod. } p),$$

ou bien

$$2^{p-2} V_{p-1} \equiv \frac{P^{p+1} - \Delta^{\frac{p+1}{2}}}{P^2 - \Delta}, \quad (\text{Mod. } p).$$

Mais on a, dans le cas présent

$$P^{p+1} \equiv P^2 \quad \text{et} \quad \Delta^{\frac{p+1}{2}} \equiv \Delta, \quad (\text{Mod. } p);$$

donc

$$2^{p-2} V_{p-1} \equiv 1, \quad (\text{Mod. } p),$$

et finalement, en multipliant par 2 et appliquant le théorème de FERMAT :

(139) $$V_{p-1} \equiv 2, \quad (\text{Mod. } p).$$

CINQUIÈME CAS. Δ *est non-résidu quadratique de p.*

On a, comme précédemment,

$$2^p U_{p+1} = \frac{p+1}{1} P^p + \frac{(p+1)p(p-1)}{1.2.3} P^{p-2}\Delta + \ldots\ldots + \frac{p+1}{1} P\Delta^{\frac{p-1}{2}},$$

$$2^p V_{p+1} = P^{p+1} + \frac{(p+1)p}{1.2} P^{p-1}\Delta + \ldots\ldots + \Delta^{\frac{p+1}{2}},$$

et, puisque p est premier,

$$2 U_{p+1} \equiv P(1 + \Delta^{\frac{p-1}{2}}),$$

$$2 V_{p+1} \equiv P^2 + \Delta . \Delta^{\frac{p-1}{2}}.$$

Mais, par hypothèse Δ est non-résidu quadratique de p, et, par suite

$$\Delta^{\frac{p-1}{2}} \equiv -1, \quad (\text{Mod. } p);$$

on a donc

(140) $$U_{p+1} \equiv 0, \quad V_{p+1} \equiv 2Q, \quad (\text{Mod. } p);$$

de là, cette proposition :

THÉORÈME : *Dans les séries récurrentes U_n de seconde et de troisième espèce, tout nombre premier p, dont Δ est un non-résidu quadratique, divise U_{p+1}.*

Désignons encore par ω le rang d'arrivée du nombre premier p dans la série des U_n, et par k un nombre entier quelconque ; on a

$$U_{k\omega} \equiv 0, \quad (\text{Mod. } p);$$

par conséquent, si p n'est pas diviseur de Q ou de Δ, on a

$$k_0\omega = p \mp 1,$$

en prenant le signe — ou le signe + suivant que Δ est résidu ou non-résidu de p ; on en déduit

$$p = k_0\omega \pm 1,$$

et, par conséquent :

THÉORÈME : *Dans les séries récurrentes de seconde espèce, les diviseurs propres de U_ω sont de la forme linéaire $p = k\omega \pm 1$, suivant que Δ est résidu ou non-résidu de p.*

En suivant une marche analogue à celle que nous avons suivie dans le paragraphe précédent, on obtient par la considération des diviseurs de $U_p\lambda$, le théorème suivant.

THÉORÈME : *Il y a une série indéfinie de diviseurs premiers communs aux formes quadratiques $x^2 - Qy^2$ et $x_1^2 - py_1^2$, lorsque p désigne un nombre premier de la forme $4q+1$; et une série indéfinie de diviseurs communs aux deux formes $x^2 - Qy^2$ et $\Delta x_1^2 + py_1^2$, lorsque p désigne un nombre premier de la forme $4q+3$.*

Nous appliquerons les résultats qui précédent, aux séries de FIBONACCI et de PELL. Pour la première, on a $P = 1$, $Q = -1$, et $\Delta = 5$, d'autre part, on sait,* que le nombre 5 est résidu de tous les nombres premiers qui sont résidus de 5, et non-résidus de tous les nombres premiers impairs qui sont non-résidus de 5 lui-même. Par conséquent :

Dans la série de FIBONACCI, *tout nombre p premier impair, de la forme $10q \pm 1$, divise le terme de rang $p-1$, et tout nombre p premier impair de la forme $10q \pm 3$ divise le terme de rang $p+1$.*

D'ailleurs, les nombres 2 et 5 divisent respectivement les termes dont le rang est un multiple de 3 ou de 5.

* GAUSS.—*Disquisitiones Arithmeticæ.* Nos. 121, 122 et 123.

Pour la série de PELL, $P = 2$, $Q = -1$, $\Delta = 2^2 \times 2$; d'autre part, on sait que le nombre 2 est résidu de tout nombre qui n'est pas divisible par 4, ni par aucun nombre premier de la forme $8q + 3$ ou $8q + 5$, et non-résidu de tous les autres; par conséquent:

Dans la série de PELL, *tout nombre premier p de la forme* $8q \pm 1$ *divise* U_{p-1}, *et tout nombre premier p de la forme* $8q \pm 3$ *divise* U_{p+1}.

Les théorèmes que nous venons de démontrer conduisent à la décomposition des termes des séries récurrentes de seconde et de troisième espèce, en facteurs premiers. On a ainsi, par exemple, dans la série de FIBONACCI:

$$U_{41} = 1655\ 80141 = 2789 \times 59369,$$
$$U_{53} = 5\ 33162\ 91173 = 953 \times 559\ 45741,$$
$$U_{59} = 95\ 67220\ 26041 = 353 \times 27102\ 60697.$$

Nous ajouterons une remarque importante dont on retrouve l'origine dans la correspondance de FERMAT, mais seulement pour les séries de première espèce.

Soit encore, par exemple, la série de FIBONACCI; les nombres premiers p, des formes linéaires $20q + 13$ et $20q + 17$, divisent U_{p+1}, et l'on a

$$p + 1 = 20q + 14 \text{ ou } p + 1 = 20q + 18,$$

et aussi

$$U_{20q+14} = U_{10q+7} V_{10q+7}, \text{ et } U_{20q+18} = U_{10q+9} V_{10q+9};$$

mais, d'autre part, les diviseurs de V_{2n+1} appartiennent aux formes linéaires $20q + 1, 9, 11, 19$; par conséquent, les nombres premiers de la forme $20q + 13$ ou $20q + 17$ divisent respectivement U_{10q+7} et U_{10q+9}, et disparaissent de la série des V_n qui ne contient donc pas tous les nombres premiers. En appliquant ce raisonnement aux séries de FERMAT et de PELL, on en déduit les principes suivants:

Dans la série de FIBONACCI, *les termes* V_n *ne contiennent aucun nombre premier des formes linéaires* $20q + 13$, $20q + 17$.

Dans la série de FERMAT, *les termes* V_n *ne contiennent aucun nombre premier de la forme* $8q + 7$.

Dans la série de PELL, *les termes* V_n *ne contiennent aucun nombre premier de la forme* $8q + 5$.

Nous donnons dans le tableau de la page 299, la décomposition en facteurs premiers des termes de la série de FIBONACCI, limitée aux soixante premiers termes.

TABLEAU DES FACTEURS PREMIERS DE LA SÉRIE RÉCURRENTE DE LÉONARD DE PISE.

n.	u_n.	Div. impropres.	Div. propres.	n.	u_n.	Diviseurs impropres.	Diviseurs propres.
1	1	—	1.	31	13 46269	—	557 $\times$ 2417.
2	1	—	1.	32	21 78309	3 $\times$ 7 $\times$ 47.	2207.
3	2	—	2.	33	35 24578	2 $\times$ 89.	19801.
4	3	—	3.	34	57 02887	1597.	3571.
5	5	—	5.	35	92 27465	5 $\times$ 13.	1 41961.
6	8	2^3.	—	36	149 30352	$2^4 \times 3^3 \times 17 \times 19$.	107.
7	13	—	13.	37	211 57817	—	73 $\times$ 149 $\times$ 2221.
8	21	3.	7.	38	390 88169	37 $\times$ 113.	9349.
9	34	2.	17.	39	632 45986	2 $\times$ 233.	1 35721.
10	55	5.	11.	40	1023 34155	3 $\times$ 5 $\times$ 7 $\times$ 11 $\times$ 41.	2161.
11	89	—	89.	41	1655 80141	—	2789 $\times$ 59369.
12	144	$2^4 \times 3^2$.	—	42	2679 14296	$2^3 \times 13 \times 29 \times 421$.	211.
13	233	—	233.	43	4334 94437	—	4334 94437.
14	377	13.	29.	44	7014 08733	3 $\times$ 89 $\times$ 199.	43 $\times$ 307.
15	610	2 $\times$ 5.	61.	45	11349 03170	2 $\times$ 5 $\times$ 17 $\times$ 61.	1 09441.
16	987	3 $\times$ 7.	47.	46	18363 11903	28657.	139 $\times$ 461.
17	1597	—	1597.	47	29712 15073	—	29712 15073.
18	2584	$2^3 \times 17$.	19.	48	48075 26976	$2^6 \times 3^2 \times 7 \times 23 \times 47$.	1103.
19	4181	—	37 $\times$ 113	49	77787 42049	13.	97 $\times$ 61 68709.
20	6765	3 $\times$ 5 $\times$ 11.	41.	50	1 25862 69025	$5^2 \times 11 \times 3001$.	101 $\times$ 151.
21	10946	2 $\times$ 13	421.	51	2 03650 11074	2 $\times$ 1597.	63 76021.
22	17711	89.	199.	52	3 29512 80099	3 $\times$ 233 $\times$ 521.	90481.
23	28657	—	28657.	53	5 33162 91173	—	953 $\times$ 559 45741.
24	46368	$2^4 \times 3^2 \times 7$.	23.	54	8 62675 71272	$2^3 \times 17 \times 19 \times 53 \times 109$.	5779.
25	75025	5^2.	3001.	55	13 95838 62445	5 $\times$ 89.	661 $\times$ 4 74541
26	1 21393	233.	521.	56	22 58514 33717	$3 \times 7^2 \times 13 \times 29 \times 281$.	14503.
27	1 96418	2 $\times$ 17.	53 $\times$ 109.	57	36 54352 96162	2 $\times$ 37 $\times$ 113.	43 71901.
28	3 17811	3 $\times$ 13 $\times$ 29.	281.	58	59 12867 29879	5 14229.	59 $\times$ 19489.
29	5 14229	—	5 14229.	59	95 67220 26041	—	353 $\times$ 27102 60697.
30	8 32040	$2^3 \times 5 \times 11 \times 61$.	31.	60	154 80087 55920	$2^4 \times 3^2 \times 5 \times 11 \times 31 \times 41 \times 61$.	2521.

SECTION XXVI.

Sur la périodicité des fonctions numériques et sur la géneralisation du CANON ARITHMETICUS.

Les résultats développés dans les deux sections précédentes, conduisent immédiatement à la périodicité numérique des fonctions que nous étudions ici, par la considération de leurs résidus suivant un module premier p ou suivant un module quelconque m. Cette question a été présentée sous une forme différente, et seulement pour les séries de première espèce, par GAUSS, dans les *Disquisitiones Arithmeticæ*, sous le nom de *théorie des indices*, et développée par JACOBI dans le *Canon Arithmeticus*. Tous ces résultats peuvent être résumés et généralisés, dans le théorème fondamental suivant, qui contient une extension du *Théorème de* FERMAT *généralisé* par EULER.

THÉORÈME FONDAMENTAL: *Si l'on désigne par m un nomber premier avec le produit des racines d'une équation du second degré à coefficients commensurables,*

$$m = p^\pi r^\rho s^\sigma \ldots\ldots,$$

par Δ *le discriminant de l'équation, et par* $\left(\frac{\Delta}{p}\right)$ *le reste de la division de* $\Delta^{\frac{p-1}{2}}$ *par* p, *et égal à* $+1$ *ou à* -1, *suivant que* Δ *est résidu quadratique, ou non-résidu quadratique de* p; *et, soit, de plus*

$$\psi(m) = p^{\pi-1} r^{\rho-1} s^{\sigma-1} \ldots \left[p - \left(\frac{\Delta}{p}\right)\right]\left[r - \left(\frac{\Delta}{r}\right)\right]\left[s - \left(\frac{\Delta}{s}\right)\right]\ldots;$$

on a la congruence

(142) $$U_{\psi(m)} \equiv 0, \quad (\text{Mod. } m).$$

Réciproquement, si U_n *est divisible par* m, *le nombre* n *est un multiple quelconque d'un certain diviseur* μ *de* $\psi(m)$.

Ce nombre μ est, par extension, l'exposant auquel appartient a ou b par rapport au module m; on retrouve le théorème d'EULER, en supposant $b = 1$.

Quant à la périodicité numérique des résidus, elle résulte des formules d'addition. On a d'abord, en faisant $n = k\omega$ dans les formules (49),

$$2U_{m+k\omega} = U_m V_{k\omega} + U_{k\omega} V_m,$$
$$2V_{m+k\omega} = V_m V_{k\omega} + \Delta U_m U_{k\omega};$$

par conséquent, si ω désigne le rang d'arrivée du nombre premier p dans la série des U_n, on a

(143) $$\left.\begin{aligned} 2U_{m+k\omega} &\equiv V_{k\omega} U_m, \\ 2V_{m+k\omega} &\equiv V_{k\omega} V_m, \end{aligned}\right\} \quad (\text{Mod. } p).$$

Supposons d'abord qu'il s'agisse des fonctions de première espèce, ou lorsque Δ est résidu de p, des fonctions de deuxième et de troisième espèce; déterminons le nombre k de telle sorte que l'on ait

$$V_{k\omega} \equiv 2, \quad (\text{Mod. } p),$$

ce qui a lieu pour $k\omega = p - 1$, mais aussi, dans la plupart des cas, pour un certain diviseur π de $p - 1$; on aura alors, pour h entier et positif, mais quelconque, les formules

$$(144) \qquad \left.\begin{aligned} U_{m+h\pi} &\equiv U_m, \\ V_{m+h\pi} &\equiv V_m, \end{aligned}\right\} \quad (\text{Mod. } p).$$

Celles ci sont analogues aux formules qui donnent la périodicité des fonctions circulaires; leur application conduit, lorsque l'on remplace le nombre premier p par un module quelconque m, et que l'on tient compte de la *loi de répétition*, à des formules nouvelles contenant la généralisation de résultats indiqués par ARNDT et SANCERY.*

Mais dans le cas des séries de seconde et de troisième espèce il n'en est plus absolument de même, lorsque Δ est non-résidu de p. En posant $\omega' = p + 1$, on a alors;

$$\left.\begin{aligned} U_{m+\omega'} &\equiv Q\,U_m, \\ V_{m+\omega'} &\equiv Q\,V_m, \end{aligned}\right\} \quad (\text{Mod. } p),$$

et, plus généralement, pour k entier et positif,

$$(145) \qquad \left.\begin{aligned} U_{m+k\omega'} &\equiv Q^k\,U_m, \\ V_{m+k\omega'} &\equiv Q^k\,V_m, \end{aligned}\right\} \quad (\text{Mod. } p);$$

par conséquent, si μ désigne l'exposant auquel appartient Q suivant le module p, on aura

$$\left.\begin{aligned} U_{m+k\mu\omega'} &\equiv U_m, \\ V_{m+k\mu\omega'} &\equiv V_m, \end{aligned}\right\} \quad (\text{Mod. } p).$$

Ainsi dans ce dernier cas, l'amplitude de la période est égale à $\mu\omega'$.

SECTION XXVII.

Sur l'inversion du théorème de FERMAT *et sur la vérification des grands nombres premiers.*

On sait que le théorème de WILSON qui consiste, pour p premier, dans la congruence

$$1 \, . \, 2 \, . \, 3 \, . \, . \, . \, . \, . \, (p - 1) \equiv -1, \quad (\text{Mod. } p),$$

* *Journal de Crelle*, t. XXXI; pag. 260 et suiv. 1846.—*Bulletin de la Société Mathématique de France*, t. IV. pag. 17 et suiv. Paris, 1876.

s'applique exclusivement aux nombres premiers, et donne, par suite, un procédé théorique, mais illusoire dans la pratique, pour reconnaître si un nombre donné est premier. Il n'en est pas de même du théorème de FERMAT. En désignant par a un nombre inférieur à p, on a

$$a^{p-1} \equiv 1, \quad (\text{Mod. } p);$$

mais ce théorème n'est pas restreint aux nombres premiers, et cette congruence peut être vérifiée pour des modules composés; ainsi, on a, par exemple,

$$2^{37 \times 73 - 1} \equiv 1, \quad (\text{Mod. } 37 \times 73).$$

Cependant, on peut énoncer le théorème suivant que l'on doit considérer comme la proposition réciproque de celle de FERMAT.

THÉORÈME: *Si $a^x - 1$ est divisible par p, lorsque $x = p - 1$, et n'est pas divisible par p, pour x inférieur à $p - 1$, le nombre p est premier.*

On sait que, dans ce cas, a est une *racine primitive de p*; de plus, il est facile de voir que si $p - 1$ est égal à une puissance de puissance de 2, a est non-résidu quadratique de p. Ce théorème rentre dans le suivant, dont la démonstration résulte immédiatement des propriétés des fonctions numériques simplement périodiques, et s'applique aux trois espèces de séries:

THÉORÈME FONDAMENTAL: *Si dans l'une des séries récurrentes U_n, le terme U_{p-1} est divisible par p, sans qu'aucun des termes de la série dont le rang est un diviseur de $p - 1$ le soit, le nombre p est premier; de même si U_{p+1} est divisible par p, sans qu'aucun des termes de la série dont le rang est un diviseur de $p + 1$ le soit, le nombre p est premier.*

En effet, puisque p divise $U_{p \pm 1}$, tous les termes divisibles par p ont un rang égal à un multiple quelconque d'un certain diviseur de $p \pm 1$; d'autre part, supposons p non premier et égal, par exemple, au produit de deux nombres premiers r et s, on a

$$U_{r \pm 1} \equiv 0, \quad (\text{Mod. } r), \qquad U_{s \pm 1} \equiv 0, \quad (\text{Mod. } s),$$

et, par suite le terme dont le rang est $(r \pm 1)(s \pm 1)$ est divisible par rs; mais, par hypothèse p divise le terme de rang $rs \pm 1$, et, par conésquent aussi, le terme dont le rang est égal à la différence des précédents, c'est-à-dire

$$(r \pm 1)(s \pm 1) - (rs \pm 1),$$

ou bien

$$\pm r \pm s \pm 1 \pm 1.$$

Mais ce dernier nombre est évidemment plus petit que rs; par conséquent, si p n'est pas premier, il divise un terme dont le rang est inférieur à $p \pm 1$; c'est ce que ne suppose pas l'énoncé.

On obtiendrait le même résultat en supposant p égal à un nombre impair quelconque, en faisant voir (Section XXVI, Théor. fond.) que

$$m \pm 1 - \psi(m)$$

est plus petit que $m \pm 1$.

Dans l'application de ce théorème, on calcule les termes dont le rang est un diviseur quelconque de $p \pm 1$, au moyen des formules d'addition et de multiplication des fonctions numériques, que nous avons exposées ci-dessus. Nous donnerons d'abord un exemple numérique très-simple.

EXEMPLE : Soit

$$2^7 - 1 = 127.$$

Pour savoir si 127 est premier, nous calculons U_{128} dans la série de FIBONACCI; on a alors les formules

$$V_{4n+2} = V_{2n+1}^2 + 2, \qquad V_{4n} = V_{2n}^2 - 2;$$

on forme ainsi le tableau

$$\begin{aligned}
U_4 &= U_2\,(V_1^2 + 2) = U_2 \times 3,\\
U_8 &= U_4\,(V_2^2 - 2) = U_4 \times 7,\\
U_{16} &= U_8\,(V_4^2 - 2) = U_8 \times 47,\\
U_{32} &= U_{16}\,(V_8^2 - 2) = U_{16} \times 2207,\\
U_{64} &= U_{32}\,(V_{16}^2 - 2) = U_{32} \times 48\,70847,\\
U_{128} &= U_{64}\,(V_{32}^2 - 2) = U_{64} \times 2732\,51504\,97407.
\end{aligned}$$

Or 127 divise le dernier facteur et ne divise aucun des précédents, ainsi $2732\,51504\,97407 = 127 \times 18\,68122\,08641$, par conséquent 127 est un nombre premier. On simplifie considérablement le calcul par la méthode des congruences, en remplaçant continuellement les nombres $V_2, V_4, V_8, \ldots$ par leurs résidus suivant le module 127. En tenant compte de cette observation, le tableau précédent devient :

$$\left.\begin{aligned}
V_4 &= 3^2 - 2 = 7,\\
V_8 &= 7^2 - 2 = 47,\\
V_{16} &= 47^2 - 2 \equiv 48,\\
V_{32} &\equiv 48^2 - 2 \equiv 16,\\
V_{64} &\equiv 16^2 - 2 \equiv 0.
\end{aligned}\right\}\quad (\text{Mod. } 127).$$

Cette méthode de vérification des grands nombres premiers, qui repose sur le principe que nous venons de démontrer, est la *seule méthode directe et pratique*, connue actuellement, pour résoudre le problème en question ; elle est opposée, pour ainsi dire, à la méthode de vérification d'EULER, déduite de la consi-

dération des résidus potentiels. Dans celle-ci, on divise le nombre soupçonné premier, par des nombres inférieurs à sa racine carrée, et qui appartiennent à des formes linéaires déterminées que l'on doit d'abord calculer; *le dividende est constant, et le diviseur variable*, mais inférieur, il est vrai, au nombre essayé; c'est *l'insuccès* de ces divisions dont le nombre est considérable, malgré la forme linéaire du diviseur, qui conduit à affirmer que le nombre essayé est premier. Dans notre méthode, au contraire, on divise, par le nombre soupçonné premier, des nombres d'un calcul facile, obtenus par la multiplication des fonctions numériques; ici le *dividende est variable et le diviseur constant;* par conséquent, on remplace les divisions par de simples soustractions, si l'on a calculé préalablement les dix premiers multiples de ce diviseur constant; en outre, le nombre des opérations est peu considérable; c'est le *succès* de l'opération qui conduit à affirmer que le nombre essayé est premier. Ainsi, en cas de réussite, notre méthode est affranchie de l'incertitude des calculs numériques.

Pour vérifier la dernière assertion du P. MERSENNE, sur le nombre supposé premier

$$2^{257} - 1,$$

et qui a *soixante-dix-huit* chiffres, il faudrait à l'humanité tout entière, formée de mille millions d'individus, calculant simultanément et sans interruption, un temps supérieur à un nombre de siècles representé par un nombre de vingt chiffres; par notre méthode, il suffit d'effectuer successivement les carrés de 250 nombres ayant 78 chiffres, au plus; cette opération ne demanderait pas, à deux calculateurs habiles contrôlant leurs opérations, plus de huit mois de travail. Nous appliquerons d'abord le théorème fondamental à la vérification des grands nombres premiers de la série de FERMAT qui appartiennent à la forme

$$p = 2^{4q+3} - 1,$$

dans laquelle nous supposerons l'exposant $4q + 3$ égal à un nombre premier tel que $8q + 7$ soit un nombre composé. En effet, si $4q + 3$ n'est pas premier, le nombre p est composé; d'autre part, nous avons démontré (Section XXIII) que si $4q + 3$ et $8q + 7$ sont premiers, le nombre p est encore composé.

En supposant p premier, on a immédiatement

$$A \equiv 2^3 - 1, \quad (\text{Mod. } 5);$$

donc, dans cette hypothèse p est non-résidu de 5, et divise le terme dont rang est égal à $p + 1$ ou à l'un des diviseurs de $p + 1$, dans la série de

FIBONACCI; mais tous ces diviseurs sont de la forme 2^λ, et pour former les termes qui correspondent à ces rangs, il suffit d'appliquer les formules de duplication des fonctions numériques. On a alors

$$U_{2^{\lambda+1}} = U_{2^\lambda} V_{2^\lambda} \quad \text{et} \quad V_{2^{\lambda+1}} = [V_{2^\lambda}]^2 - 2(-1)^{2^\lambda},$$

et l'application du théorème fondamental donne le principe suivant:

THÉORÈME II: *Soit le nombre* $p = 2^{4q+3} - 1$ *pour lequel* $4q + 3$ *est premier, et* $8q + 7$ *composé; on forme la série* r_n

$$1, 3, 7, 47, 2207, \ldots\ldots$$

par la relation, pour $n > 1$,

$$r_{n+1} = r_n^2 - 2;$$

le nombre p *est premier lorsque le rang du premier terme, divisible par* p, *occupe un rang compris entre* $2q + 1$ *et* $4q + 2$; *le nombre* p *est composé, si aucun des* $4q + 2$ *premiers termes de la série n'est divisible par* p; *enfin, si* α *désigne le rang du premier terme divisible par* p, *les diviseurs de* p *appartiennent à la forme linéaire* $2^\alpha K \pm 1$, *combinée avec celles des diviseurs de* $x^2 - 2y^2$.

Dans la pratique, on calcule par congruences, en ne conservant que les résidus suivant le module p, ainsi que nous l'avons montré précédemment pour le nombre $p = 2^7 - 1$. Nous avons indiqué un autre procédé de calcul, qui repose sur l'emploi du système de numération binaire, et qui conduit à la construction d'un mécanisme propre à la vérification des grands nombres premiers.

Dans ce système de numération, la multiplication consiste simplement dans le déplacement longitudinal du multiplicande; d'autre part, il est clair que le reste de la division de 2^m par $2^n - 1$ est égal à 2^r, r désignant le reste de la division de m par n; par conséquent dans l'essai de $2^{31} - 1$, par exemple, il suffira d'opérer sur des nombres ayant, au plus, 31 chiffres. Le tableau de la page 306 donne le calcul du résidu de $V_{2^{26}}$ déduit du résidu de $V_{2^{25}}$ suivant le module $2^{31} - 1$, par la formule

$$V_{2^{26}} \equiv (V_{2^{25}})^2 - 2, \quad (\text{Mod. } 2^{31} - 1);$$

les carrés noirs représentent les unités des différents ordres du système binaire, et les carrés blancs représentent les zéros. La première ligne est le résidu de $V_{2^{25}}$; les 31 premières lignes numérotées 0 — 30 figurent le carré de $V_{2^{25}}$; les 4 lignes numérotées 0, 1, 2, 3 du bas de la page indiquent l'addition des unités de chaque colonne, avec les reports; on a retranché une unité de la première colonne à gauche; enfin la dernière ligne est le résidu de $V_{2^{26}}$.

	30	29	28	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	
0	■	□	□	■	■	■	■	□	■	□	□	■	■	■	□	□	□	■	■	□	■	■	■	□	■	■	□	■	■	□	■	0
1	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	1
2	□	■	■	■	■	□	■	□	□	■	■	■	□	□	□	■	■	□	■	■	■	□	■	■	□	■	■	□	■	■	□	2
3	■	■	■	■	□	■	□	□	■	■	■	□	□	□	■	■	□	■	■	■	□	■	■	□	■	■	□	■	■	□	□	3
4	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	4
5	■	■	□	■	□	□	■	■	■	□	□	□	■	■	□	■	■	■	□	■	■	□	■	■	□	■	■	□	□	■	■	5
6	■	□	■	□	□	■	■	■	□	□	□	■	■	□	■	■	■	□	■	■	□	■	■	□	■	■	□	□	■	■	■	6
7	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	7
8	■	□	□	■	■	■	□	□	□	■	■	□	■	■	■	□	■	■	□	■	■	□	■	■	□	□	■	■	■	■	□	8
9	□	□	■	■	■	□	□	□	■	■	□	■	■	■	□	■	■	□	■	■	□	■	■	□	□	■	■	■	■	□	■	9
10	□	■	■	■	□	□	□	■	■	□	■	■	■	□	■	■	□	■	■	□	■	■	□	□	■	■	■	■	□	■	□	10
11	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	11
12	■	■	□	□	□	■	■	□	■	■	■	□	■	■	□	■	■	□	■	■	□	□	■	■	■	■	□	■	□	□	■	12
13	■	□	□	□	■	■	□	■	■	■	□	■	■	□	■	■	□	■	■	□	□	■	■	■	■	□	■	□	□	■	■	13
14	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	14
15	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	15
16	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	16
17	■	■	□	■	■	■	□	■	■	□	■	■	□	■	■	□	□	■	■	■	■	□	■	□	□	■	■	■	□	□	□	17
18	■	□	■	■	■	□	■	■	□	■	■	□	■	■	□	□	■	■	■	■	□	■	□	□	■	■	■	□	□	□	■	18
19	□	■	■	■	□	■	■	□	■	■	□	■	■	□	□	■	■	■	■	□	■	□	□	■	■	■	□	□	□	■	■	19
20	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	20
21	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	21
22	■	□	■	■	□	■	■	□	■	■	□	□	■	■	■	■	□	■	□	□	■	■	■	□	□	□	■	■	□	■	■	22
23	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	23
24	■	■	□	■	■	□	■	■	□	□	■	■	■	■	□	■	□	□	■	■	■	□	□	□	■	■	□	■	■	■	□	24
25	■	□	■	■	□	■	■	□	□	■	■	■	■	□	■	□	□	■	■	■	□	□	□	■	■	□	■	■	■	□	■	25
26	□	■	■	□	■	■	□	□	■	■	■	■	□	■	□	□	■	■	■	□	□	□	■	■	□	■	■	■	□	■	■	26
27	■	■	□	■	■	□	□	■	■	■	■	□	■	□	□	■	■	■	□	□	□	■	■	□	■	■	■	□	■	■	□	27
28	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	28
29	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	29
30	■	■	□	□	■	■	■	■	□	■	□	□	■	■	■	□	□	□	■	■	□	■	■	■	□	■	■	□	■	■	□	30
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CALCUL DU RÉSIDU DE $V_{2^{26}}$ AU MOYEN DE $V_{2^{25}}$ SUIVANT LE MODULE $2^{31} - 1$.

Le tableau de la page 307 contient l'ensemble de tous les résidus de V_2, V_{2^2}, V_{2^3}, ... $V_{2^{29}}$, $V_{2^{30}}$ suivant le module $2^{31} - 1$. La dernière ligne, entièrement composée de zéros, nous montre que $2^{31} - 1$ est premier.

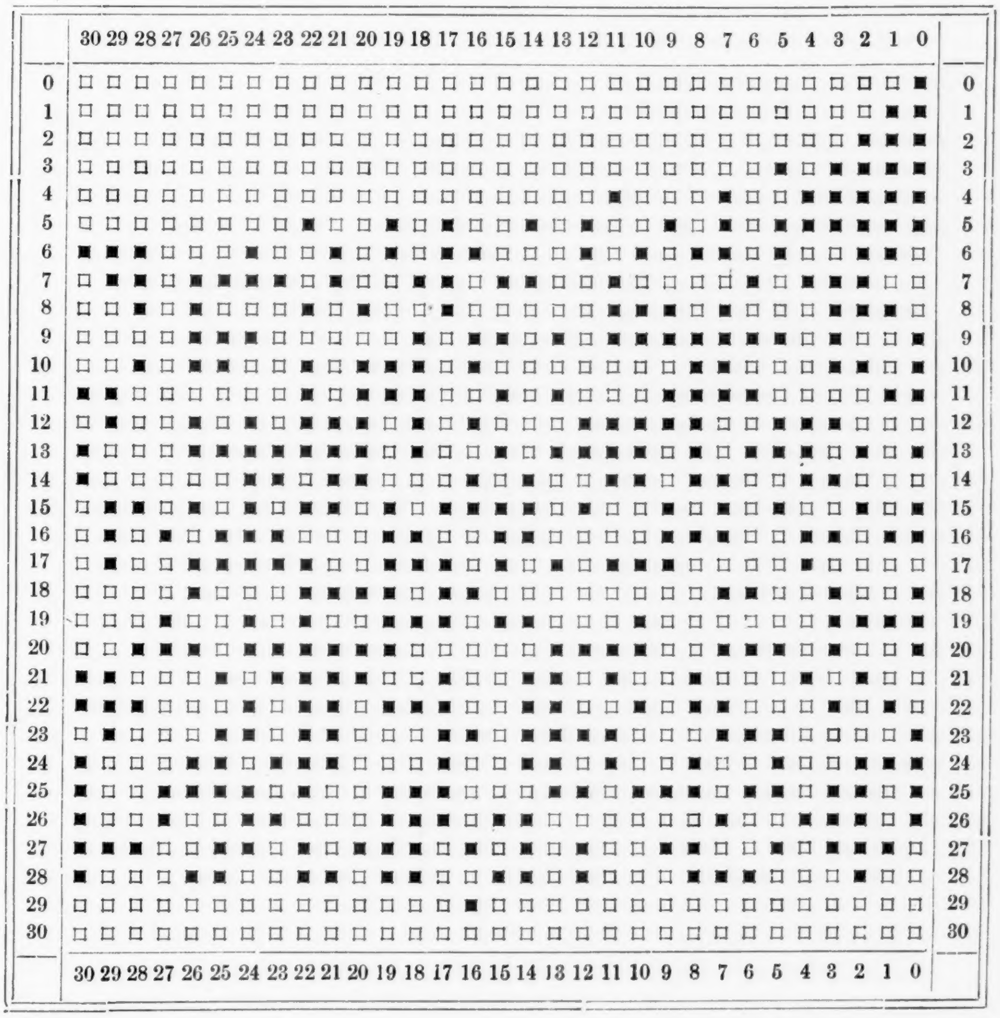

DIAGRAMME DU NOMBRE PREMIER $2^{31}-1$.

Ce tableau est, en quelque sorte, un fragment du *Canon Arithmeticus*, correspondant au nombre premier $2^{31}-1$ pour la racine primitive $\dfrac{1 \pm \sqrt{5}}{2}$.

On pourrait ainsi construire les *diagrammes* des nombres premiers de la forme $2^{4q+3}-1$. Nous donnons aussi celui du nombre $2^{19}-1$; nous espérons donner ultérieurement ceux des nombres $2^{67}-1$ et $2^{127}-1$.

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DIAGRAMME DU NOMBRE PREMIER $2^{19}-1$.

Lorsque l'on aura, par l'application du théorème fondamental, à vérifier de grands nombres premiers de la forme $\dfrac{3^n \pm 1}{3 \pm 1}$, on emploiera aussi avec succès le système ternaire, dans lequel on se servira seulement des chiffres 0, 1 et $\bar{1}$, à caractéristiques positives on négatives. Pour la vérification des grands nombres de la forme $10^{2^n}+1$, on se servira facilement du système décimal, pour le calcul des résidus.

Lorsque le nombre essayé n'est pas premier, nous avons vu qu'on ne trouvera aucun résidu nul. Soit, par exemple, le nombre $p = 2^{11} - 1 = 2047$; les résidus que nous considérons sont, dans ce cas,

$$1, 3, 7, 47, 160, 1034, 620, -438, -576, 160, \ldots,$$

et se reproduisent périodiquement à partir de 160 suivant les cinq résidus

$$160, 1034, 620, -438, -576.$$

On peut donner une autre forme d'énoncé, au Théorème II, et aux suivants, en tenant compte des formules qui concernent les radicaux continus (Section XV); on a, par exemple:

THÉORÈME III: *Pour que le nombre* $p = 2^{4q+3} - 1$ *soit premier, il faut et il suffit que la congruence* $3 \equiv 2 \cos \frac{\pi}{2^{2q+1}}$, (Mod. p),

soit vérifiée, après la disparition successive des radicaux contenus dans la valeur du cosinus. Nous démontrerons ultérieurement que cette condition est nécessaire et suffisante.

On observera encore que les nombres de la série

$$1, 3, 7, 47, 2207, \ldots\ldots$$

appartiennent tous, à partir du troisième, à la forme linéaire $5q + 2$; mais, d'autre part (Section VIII, Théor. II), les diviseurs de ces nombres appartiennent aux formes linéaires

$$20q + 1, 3, 7, 9;$$

par conséquent, chacun des termes de la série précédente contient un diviseur premier de la forme $5q + 2$; il en résulte immédiatement cette proposition:

THÉORÈME IV: *Il y a une infinité de nombres premiers appartenant à la forme linéaire* $5q + 2$.

On voit encore que les nombres de la série ont la forme $8h + 7$; mais, d'autre part, la forme des diviseurs quadratiques indique que les diviseurs de ces nombres sont de l'une des formes $8h + 1$, ou $8h + 7$; par conséquent, chacun des nombres de la série contient au moins un diviseur de la forme $8h + 7$, et, par suite:

THÉORÈME V: *Il y a une infinité de nombres premiers appartenant à la forme linéaire* $8h + 7$.

Les théorèmes suivants permettent d'arriver à un grand nombre de théorèmes analogues, qui sont des cas particuliers du théorème fondamental de LEJEUNE-DIRICHLET, sur la progression arithmétique. Nous devons observer, cependant, que les nombres de la forme $5q + 2$ ne sont pas tous compris dans la série que nous considérons ici, et qu'il en est de même dans tous les autres cas. Ainsi les théorèmes précédents diffèrent, au fond, des cas analogues de la progression arithmétique. La méthode que nous employons s'applique d'ailleurs, très-facilement, à la démonstration du théorème général suivant:

THÉORÈME VI: *Si A et Q désignent deux nombres quelconques premiers entre eux, la série*

$$r_0, r_1, r_2, r_3, \ldots\ldots r_n,$$

dans laquelle on a

$$r_0 = A\,, \quad r_1 = A^2 + 2Q\,, \quad r_{n+1} = r_n^2 - 2Q^{2^n}\,,$$

contient comme diviseurs, des nombres premiers tous différents.

Les formules de multiplication des fonctions numériques conduisent à des résultats analogues.

Par des considérations semblables aux précédentes, on démontrera les théorèmes suivants.

THÉORÈME VII: *Soit le nombre* $p = A\,.\,2^q - 1$, *et*

$$\left.\begin{array}{ll} 1^\circ, & q \equiv 0\,, \\ 2^\circ, & q \equiv 1\,, \\ 3^\circ, & q \equiv 2\,, \\ 4^\circ, & q \equiv 3\,, \end{array}\right\} (\text{Mod. } 4)\,, \text{ et } \left.\begin{array}{lcl} A \equiv 3\,, & & \equiv 9\,, \\ A \equiv 7\,, & \text{ou} & \equiv 9\,, \\ A \equiv 1\,, & & \equiv 7\,, \\ A \equiv 1\,, & & \equiv 3\,, \end{array}\right\} (\text{Mod. } 10)\,;$$

on forme les q premiers termes de la série

$$r_1\,, r_2\,, r_3\,, r_4\,, \ldots\,,$$

par la relation de récurrence

$$r_{n+1} = r_n^2 - 2\,,$$

en prenant pour r_1 *et* r_2 *les termes* U_A *et* V_A, *de la série de* FIBONACCI. *Le nombre p est premier, lorsque le rang du premier terme divisible par p est égal à q. Si α désigne le rang du premier terme divisible par p, les diviseurs de p sont de la forme* $2^\alpha\,.\,A\,.\,k \pm 1$, *combinée avec celle des diviseurs de* $x^2 - 2y^2$ *et de* $x^2 - 2Ay^2$.

THÉORÈME VIII: *On obtient un théorème semblable en prenant*

$$p = A\,.\,2^q + 1\,,$$

avec les valeurs

$$\left.\begin{array}{ll} 1^\circ, & q \equiv 0\,, \\ 2^\circ, & q \equiv 1\,, \\ 3^\circ, & q \equiv 2\,, \\ 4^\circ, & q \equiv 3\,, \end{array}\right\} (\text{Mod. } 4)\,, \text{ et } \left.\begin{array}{lcl} A \equiv 5\,, & & \equiv 3\,, \\ A \equiv 5\,, & \text{ou} & \equiv 9\,, \\ A \equiv 5\,, & & \equiv 7\,, \\ A \equiv 5\,, & & \equiv 1\,, \end{array}\right\} (\text{Mod. } 10)\,;$$

soit, par exemple, $p = 3\,.\,2^{11} - 1 = 6143$. On forme la série des résidus

4, 18, 322, — 749, 1986, 388, 3110, 3016, 4614, 499, 0;

donc $p = 6143$ est premier.

THÉORÈME IX: *Soit le nombre*

$$p = A\,.\,3^q - 1\,,$$

avec les valeurs

$$\left.\begin{array}{ll} 1^\circ, & q \equiv 0\,, \\ 2^\circ, & q \equiv 1\,, \\ 3^\circ, & q \equiv 2\,, \\ 4^\circ, & q \equiv 3\,, \end{array}\right\} (\text{Mod. } 4)\,, \text{ et } \left.\begin{array}{lcl} A \equiv 4\,, & & \equiv 8\,, \\ A \equiv 6\,, & \text{ou} & \equiv 8\,, \\ A \equiv 2\,, & & \equiv 6\,, \\ A \equiv 2\,, & & \equiv 4\,, \end{array}\right\} (\text{Mod. } 10)\,;$$

on forme les q premiers termes de la série

$$r_1, r_2, r_3, \ldots,$$

par la formule de récurrence

$$r_{n+1} = r_n^3 + 3r_n^2 - 3,$$

déduite des formules de triplication, avec les conditions initiales

$$r_0 = U_A, \; r_1 = \frac{U_{3A}}{U_A},$$

dans la série de FIBONACCI; *le nombre p est premier lorsque le rang du premier terme divisible par p est égal à q; si α désigne le rang du premier terme divisible par p, les diviseurs de p sont de la forme* $3^\alpha . A . k \pm 1$, *combinée avec celle des diviseurs quadratiques correspondants.*

EXEMPLE: Pour $p = 2 . 3^7 - 1$, les résidus sont

$$2, 17, 1404, 0;$$

donc $p = 4373$ est un nombre premier, puis qu'il n'a pas de diviseur inférieur à sa racine carrée.

THÉORÈME X: *On a un théorème analogue en supposant*

$$p = A . 3^q + 1,$$

avec les valeurs

$$\left.\begin{array}{l} q \equiv 0, \\ q \equiv 1, \\ q \equiv 2, \\ q \equiv 3, \end{array}\right\} \text{(Mod. 4)}, \quad \text{et} \quad \begin{array}{l} A \equiv 0, \\ A \equiv 0, \\ A \equiv 0, \\ A \equiv 0, \end{array} \quad \text{ou} \quad \left.\begin{array}{l} \equiv 8, \\ \equiv 6, \\ \equiv 2, \\ \equiv 4, \end{array}\right\} \text{(Mod. 10)};$$

et la relation de récurrence

$$r_{n+1} = r_n^3 - 3r_n^2 + 3.$$

EXEMPLE: Pour $p = 2 . 3^6 + 1$, on a les résidus

$$4, 19, -57, 569, -212, 0;$$

donc $p = 1459$ est premier.

THÉORÈME XI: *Soit le nombre*

$$p = 2A . 5^q + 1;$$

on forme la série limitée à q termes, $r_0, r_1, r_2, r_3, \ldots,$ *par la relation de récurrence*

$$r_{n+1} = r_n^5 + 5r_n^3 + 5r_n,$$

et les conditions initiales

$$r_0 = U_A, \; r_1 = U_{5A},$$

dans la série de FIBONACCI; *le nombre p est premier, lorsque le rang du premier terme divisible par p est égal à q; il est composé, si aucun des q termes n'est divisible par p; enfin, si α désigne le rang du premier résidu nul, les diviseurs premiers de p sont de l'une des formes* $2A . 5^\alpha k \pm 1$.

SECTION XXVIII.

Sur la division géométrique de la circonférence en parties égales.

Dans la section précédente, nous n'avons considéré que la vérification des nombres premiers par l'emploi de la série de FIBONACCI; il est clair que toutes les autres séries donnent lieu à de semblables théorèmes; par suite de l'indétermination laissée à la somme P et au produit Q des deux racines de l'équation fondamentale, on pourra toujours s'assurer du mode de composition d'un nombre p, lorsque l'on connaîtra l'une ou l'autre des decompositions de $p+1$ ou de $p-1$, en facteurs premiers. Nous donnerons encore l'application du théorème fondamental, aux nombres premiers dans lesquels on peut diviser géométriquement la circonférence, en parties égales.

La théorie de la division géomètrique de la circonférence, en parties égales, a été donnée par GAUSS, dans la dernière section des *Disquisitiones Arithmeticæ.* Il est convenu que cette opération ne peut être exécutée que des trois maniéres suivantes: 1° par l'emploi simultané de la règle et du compas, comme dans la construction ordinaire du décagone régulier (EUCLIDE); 2° par l'emploi du compas sans la règle (MASCHERONI); 3° par l'emploi de la double règle, sans compas, c'est-à-dire d'une règle plate dont les deux bords sont rectilignes et parallèles. Cette idée ingénieuse est due à M. DE COATPONT, colonel du génie.

GAUSS a démontré que, pour diviser géométriquement la circonférence en N parties égales, il faut et il suffit que

$$N = 2^{\mu} . a_i . a_j . a_k \ldots . ,$$

μ étant arbitraire, $a_i, a_j, a_k, \ldots .$ des nombres premiers et differents, en nombre quelconque, mais de la forme

$$a_n = 2^{2^n} + 1 .$$

On a, pour les premières valeurs de n,

$$a_0 = 3 , a_1 = 5 , a_2 = 17 , a_3 = 257 , a_4 = 65537 ;$$

mais a_5 est divisible par 641 (Section XXVI), et ne peut être compris dans l'expression de N. Il reste donc deux questions importantes à résoudre: 1° comment peut-on s'assurer que a_n est premier? 2° existe-t-il une série indéfinie de nombres premiers a_n? Nous ne répondrons, pour l'instant, qu'à la première question.

Si a_n est premier, le nombre Q est résidu quadratique de a_n; donc, dans la série de PELL, V_{a_n-1} est divisible par a_n; mais $a_n - 1$ a pour diviseurs les nombres,

$$2, 2^2, 2^3, \ldots . 2^n;$$

on a donc, par l'application du théorème fondamental, et par les formules de duplication, le théorème suivant:

THÉORÈME I: *Soit le nombre* $a_n = 2^{2^n} + 1$; *on forme la série des* $2^n - 1$ *termes,*

$$6, 34, 1154, 13\,31714, 17\,73462\,17794, \ldots,$$

tels que chacun d'eux est égal au carré du précédent diminué de deux unités; le nombre a_n *est premier, lorsque le premier terme divisible par* a_n *est compris entre les termes de rang* 2^{n-1} *et* $2^n - 1$; *il est composé, si aucun des termes de la série n'est divisible par* a_n; *enfin si* $\alpha < 2^{n-1}$ *désigne le rang du premier terme divisible par* a_n, *les diviseurs premiers de* a_n *appartiennent à la forme linéaire*

$$2^{2^{\alpha+1}} . q + 1 .$$

On obtiendrait un théorème analogue pour l'essai des grands nombres premiers de la forme $A . 2^{2^n} + 1$.

Le savant P. PÉPIN a présenté à l'Académie des Sciences de Paris (*Comptes rendus*, 6 Août 1877), un autre théorème pour reconnaître les nombres premiers a_n, qui rentre dans notre méthode générale. En effet, au lieu de nous servir de la série de PELL, nous pouvons employer beaucoup d'autres séries récurrentes, et ainsi la série récurrente de première espèce, dont les termes sont donnés par l'expression

$$U_r = \frac{a^r - b^r}{a - b},$$

dans laquelle a et b désignent deux nombres entiers arbitraires. En faisant $b = 1$ et a quelconque, on obtiendra un théorème analogue au précédent; mais si, de plus, par la loi de réciprocité des résidus quadratiques, on choisit pour a un non-résidu de a_n supposé premier, $a = 5$, par exemple, il est clair que le rang du premier résidu nul sera exactement égal à $2^n - 1$. De cette façon, la forme ambigüe donnée à l'énoncé de nos théorèmes disparaît, il est vrai, et l'on obtient alors une *condition nécessaire et suffisante* pour que a_n soit premier. Il serait facile de tenir compte de cette observation, et de donner une série de théorèmes analogues, dans la recherche de la condition nécessaire et suffisante pour qu'un nombre $2^n ap \pm 1$ soit premier, lorsque a désigne un produit de facteurs premiers donnés, et p un nombre premier arbitraire. On a, par exemple, les théorèmes suivants.

THÉORÈME II: *Lorsque* $p = 10q + 7$ *ou* $p = 10q + 9$ *est un nombre premier, le nombre* $2p - 1$ *est premier si l'on a, dans la série de* FIBONACCI,

$$U_p \equiv 0, \quad (\text{Mod. } 2p - 1),$$

et réciproquement.

THÉORÈME III: *Lorsque* $p = 4q + 3$ *est un nombre premier, le nombre* $2p + 1$ *est premier si l'on a, dans la série de* FERMAT,

$$U_p \equiv 0, \quad (\text{Mod. } 2p + 1),$$

et réciproquement.

THÉORÈME IV: *Lorsque* $p = 4q + 3$ *est un nombre premier, le nombre* $2p - 1$ *est premier si l'on a, dans le série de* PELL,

$$U_p \equiv 0, \quad (\text{Mod. } 2p - 1),$$

et réciproquement.

On doit cependant observer que si la méthode indiquée par le P. PÉPIN, conduit à une forme plus claire et plus précise de l'ènoncé, qui devient ainsi semblable à celui du théorème de WILSON, il est préférable de s'en tenir, dans l'application, à la forme que nous avons adoptée. En effet, l'application de ces théorèmes repose sur une hypothèse, celle de considérer comme premier un nombre pris arbitrairement dans une certaine forme; il est plus probable de supposer, au contraire, le nombre comme composé, ainsi que semble l'indiquer l'assertion du P. MERSENNE. Par conséquent, au lieu de reculer la vérification, jusqu'à l'extrême limite, par l'emploi des non-résidus quadratiques, il serait plus pratique, dans l'exemple, de se servir de l'un des $\phi(2^{n-1})$ nombres qui appartiennent à l'exposant 2^{n-1}, pour le module a_n supposé premier; mais cette recherche directe est fort difficile. On s'assurera cependant que, par le théorème I, il suffit, pour démontrer que a_2, a_3, a_4, sont premiers, d'exécuter respectivement 3, 6, 12, opérations au lieu du nombre 4, 8, 16, qui lui correspond dans l'autre méthode.

SECTION XXIX.

Sur la vérification de l'assertion du P. MERSENNE.

Nous avons indiqué la marche à suivre pour les nombres de la forme $2^{4q+3} - 1$; il nous reste à indiquer une marche analogue pour les nombres de la forme $p = 2^{4q+1} - 1$, tels que

$$2^{61} - 1, 2^{97} - 1, \ldots, 2^{257} - 1.$$

En supposant p premier, -1 est non-résidu de p puisque p est de la forme $4k+3$, et 2 est résidu de p, puisque p est de la forme $8k+7$; donc -2 est non-résidu de p. Par conséquent, la série conjuguée de celle de PELL, c'est-à-dire la série provenant de l'équation,

$$x^2 = 2x + 3,$$

dans laquelle

$$P = 2, \quad Q = -3, \quad \Delta = 2^2 \times (-2),$$

est propre à la vérification des nombres premiers que nous considérons, puisque, si p est premier, U_{p+1} est divisible par p. Les diviseurs de $p+1$ représentent toutes les puissances de 2 jusqu'à l'exposant $4q+1$; il suffira donc de calculer les résidus de

$$U_1, \; V_1, \; V_2, \; V_4, \ldots \; V_{2^{4q}},$$

par les formules ordinaires de duplication.

Mais nous devons encore faire une observation importante, au point de vue du calcul. Puisque l'on emploie la formule

$$V_{2^{\lambda+1}} = (V_{2^\lambda})^2 - 2Q^{2^\lambda},$$

il est bon, si l'on effectue le calcul des résidus dans le système de numération décimale, de supposer $Q = \pm 1$, ou $Q = \pm 10^n$; car sans cela, on double la longueur des calculs, ainsi qu'il est facile de s'en apercevoir; si l'on opère dans le système de numération binaire, il sera commode de supposer Q égal, en valeur absolue, à l'unité ou à une puissance de 2.

Il est donc préférable d'employer la série récurrente provenant de l'équation

$$x^2 = 4x - 1,$$

dans laquelle

$$a = 2 + \sqrt{3}, \qquad b = 2 - \sqrt{3},$$

et

$$P = 4, \qquad Q = 1, \qquad \Delta = 2^2 \times 3.$$

En supposant que $p = 2^{4q+1} - 1$ est un nombre premier, on a, par la loi de réciprocité,

$$\left(\frac{3}{p}\right) = -\left(\frac{p}{3}\right),$$

puisque p et 3 sont tous deux des multiples de 4 plus 3; d'autre part, par le théorème de FERMAT

$$2^{4q+1} - 1 \equiv 1, \quad (\text{Mod. } 3);$$

donc 3 est non-résidu de p, supposé premier, et, dans ce cas, U_{p+1} est divisible par p. Les diviseurs de $p+1$ sont égaux à toutes les puissances de 2 jusqu'à $4q+1$, et, de plus, $Q = 1$.

Par conséquent, on formera la suite des résidus

$$4, 14, 194, 37634, \ldots.$$

tels que chacun d'eux est égal au carré du précédent diminué de deux unités.

EXEMPLE: Soit le nombre $2^{13} - 1 = 8191$; on trouve les résidus

$$4, 14, 194, 4870, 3953, 5970, 1857, 36, 1294, 3470, 128, 0;$$

donc le nombre $2^{13} - 1$ est premier.

On a donc le théorème suivant:

THÉORÈME: *Soit le nombre $p = 2^{4q+1} - 1$; on forme la série des résidus*

$$4, 14, 194, 37634, \ldots,$$

tels que chacun d'eux est égal au carré du précédent diminué de deux unités; le nombre p est composé, si aucun des $4q + 1$ premiers résidus n'est égal à 0; le nombre p est premier si le premier résidu nul occupe un rang compris entre $2q$ et $4q + 1$; si le rang du premier résidu est égal à $\alpha < 2q$, les diviseurs de p appartiennent à la forme linéaire

$$2^{\alpha+1}k + 1.$$

On aurait encore des théorèmes analogues pour les nombres de la forme

$$A \cdot 2^{4q+1} - 1.$$

Avant de terminer ce paragraphe, nous ferons observer que nous pensons n'avoir qu'effleuré le sujet qui nous occupe. Il reste à trouver, comme pour les nombres premiers, un criterium des nombres composés, affranchi de l'incertitude des calculs numériques; dans un grand nombre de cas, lorsque le nombre essayé n'est pas premier, il se présente une période dans la suite des résidus; mais, s'il est vrai, comme nous l'avons démontré (Section XXVI), que cette période existe, lorsque l'on considére l'ensemble des résidus de tous les termes de la série récurrente, il n'est pas démontré que cette période se manifestera, si l'on ne considére qu'un certain nombre d'entre eux, dont les rangs sont en progression géométrique. C'est là un problème important à résoudre.

En second lieu, lorsque l'ensemble des calculs démontre que le nombre essayé n'est pas premier, peut-on arriver facilement, par la connaissance de la série des rèsidus calculés, à la décomposition du nombre que l'on avait supposé premier? Ces résidus forment, comme nous l'avons dit, un fragment d'un *Canon Arithmeticus généralisé*, que l'on peut comparer aux tables des logarithmes des sinus et des cosinus, ainsi que l'on compare le Canon Arithmeticus lui-même, aux tables des logarithmes des nombres. C'est là un second problème à résoudre.

Nous avons encore indiqué (Sections IX et XXI), une première généralisation de l'idée principale de ce mémoire, dans l'étude des séries récurrentes qui naissent des fonctions symétriques des racines des équations algébriques du troisième et du quatrième degré, et, plus généralement, des racines des équations de degré quelconque, à coefficients commensurables. On trouve, en particulier, dans l'étude de la fonction.

$$U_n = \frac{\Delta\,(a^n,\, b^n,\, c^n,\, \ldots.)}{\Delta\,(a,\, b,\, c,\, \ldots.)},$$

dans laquelle a, b, c, ... désignent les racines de l'équation, et $\Delta\,(a, b, c, \ldots.)$ la *fonction alternée* des racines, ou la racine carrée du discriminant de l'équation, la généralisation des principales formules contenues dans la première partie de ce travail.

Enfin, il reste à développer la théorie de la division des fonctions numériques, et son application à l'analyse indéterminée du second degré et des degrés superieurs; c'est une étude que nous espérons publier prochainement. Nous donnons d'ailleurs, dans le dernier paragraphe qui suit, une autre généralisation des fonctions numériques périodiques, déduite de la considération des séries ordonnées suivant les puissances de la variable.

SECTION XXX.

Sur la périodicité numérique des coefficients différentiels des fonctions rationnelles d'exponentielles.

L'étude des nombres premiers contenus dans les dénominateurs des coefficients des puissances de la variable, dans les développements en séries, lorsque l'on suppose ces coefficients réduits à leur plus simple expression, a conduit EISENSTEIN à la découverte d'un théorème remarquable. En effet, ce théorème fournit un criterium qui permet de décider, à la seule inspection des facteurs premiers du dénominateur, si la fonction qui représente la somme de la série supposée convergente, est *algébrique* ou *transcendante.*

On sait encore que l'étude des facteurs premiers contenus dans les numérateurs des coefficients B_n de $\frac{z^n}{1.2.3\ldots n}$ dans le développement de $\frac{z}{1-e^z}$, ou, en d'autres termes, dans les numérateurs des *nombres de* BERNOULLI, a conduit CAUCHY, MM. GENOCCHI et KUMMER, à d'importants résultats sur la théorie des résidus quadratiques, et sur celle de l'équation indéterminée

$$x^p + y^p + z^p = 0,$$

dont FERMAT a affirmé l'impossibilité en nombres entiers, pour $p > 2$. Ainsi M. KUMMER a démontré que cette équation ne peut être vérifiée par des nombres entiers, lorsque p ne se trouve pas comme facteur dans les numérateurs des nombres de BERNOULLI B_2, B_4, B_6, ... B_{p-3}.*

En se plaçant à un point de vue différent, MM. CLAUSEN et STAUDT ont donné pour ces nombres cette expression remarquable

$$B_{2n} = A_{2n} - \frac{1}{2} - \frac{1}{\alpha} - \frac{1}{\beta} - \ldots - \frac{1}{\lambda},$$

dans laquelle A_{2n} est un nombre entier, et les dénominateurs $2, \alpha, \beta, \gamma, \ldots . \lambda$, tous les nombres premiers qui surpassent d'une unité tous les diviseurs de $2n$. Cette formule conduit au procédé le plus rapide pour le calcul de ces nombres; M. ADAMS vient de donner, par son emploi, les valeurs des 62 premiers nombres (*British Association*,—Plymouth, Août 1877.)

Nous avons indiqué aussi comment l'application combinée des théorèmes de FERMAT et de STAUDT conduit à cette propriété que les nombres

$$a\,(a^{2n} - 1)\,B_{2n},$$

sont entiers, quel que soit l'entier a. Nous allons montrer que l'étude des coefficients de $\dfrac{x^n}{1\,.\,2\,.\,3 \ldots n}$ dans le développement des fonctions rationnelles d'exponentielles, ou, en d'autres termes, les coefficients différentiels de ces fonctions, pour $x = 0$, conduit à des propriétés importantes.

On sait, en effet, que si l'on remplace x par les nombres entiers consécutifs dans la fonction

$$\phi\,(x) = Aa^x + Bb^x + Cc^x + Dd^x + \ldots .$$

dans laquelle $A, B, C, D, \ldots .$ et $a, b, c, d, \ldots .$ sont entiers, on a, pour p premier et k entier quelconque, la congruence

$$(148) \qquad \phi\,[x + k\,(p-1)] \equiv \phi\,(x), \quad (\text{Mod.}\ p).$$

Nous avons étendu cette propriété aux fonctions numériques U_n et V_n; il est facile de voir que cette proposition s'applique aux coefficients différentiels d'une fonction entière de e^x et de e^{-x}. Il nous reste à montrer que cette

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*Nous avons modifié les diverses notations qui concernent ces nombres. La présente notation se prête beaucoup plus facilement aux développements que comporte la théorie de ces nombres. Voir, sur ce sujet, les Notes insérées dans les *Comptes rendus de l'Académie des Sciences de Paris* (Septembre 1876), dans les *Annali di Matematica* (2e série, tome VIII), dans les *Nouvelles Annales de Mathématiques* (2e série, tome XVI, pag. 157), dans la *Nouvelle Correspondance Mathématique* (tome II, pag. 328, et tome III, pag. 69), dans *The Messenger of Mathematics*, (Octobre 1877), etc.

proposition s'applique encore aux coefficients différentiels d'une fonction rationnelle de e^x et de e^{-x}.

Soit d'abord

(149) $$\text{séc}\, x = 1 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \ldots;$$

M. SYLVESTER a appelé *nombres Eulériens* (*Comptes rendus*, t. LII, pag. 161), les coefficients, pris en valeur absolue, déterminés par la relation

(150) $$E_{2n} = (-1)^n\, 1, 2, 3 \ldots (2n)\, Q_{2n};$$

on a, par le changement de x en $x\sqrt{-1}$, la formule symbolique

(151) $$u = \frac{2}{e^x + e^{-x}} = e^{Ex},$$

dans laquelle on remplacera, dans le développement du second membre les exposants de E par des indices; ainsi

$$\frac{d^n u_0}{dx_0^n} = E_n.$$

En chassant les dénominateurs de l'identité (151), on obtient, par l'identification des coefficients de x^n, la formule

(152) $$(E+1)^n + (E-1)^n = 0,$$

qui permet de calculer les nombres Eulériens par voie récurrente. On a aussi le déterminant

(153) $$E_{2n} = (-1)^n \begin{vmatrix} 1 & 1 & 0 & 0 & . & . & . & . & 0 \\ 1 & 6 & 1 & 0 & . & . & . & . & 0 \\ 1 & 15 & 15 & 1 & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & 0 \\ 1 & C_{2n}^2 & C_{2n}^4 & C_{2n}^6 & . & . & . & . & C_{2n}^2 \end{vmatrix};$$

ce déterminant est formé par les lignes de rang pair et les colonnes de rang impair du triangle arithmétique. Les nombres Eulériens sont entiers et impairs; SHERK a démontré qu'ils sont terminés alternativement par les chiffres 1 et 5*. Ces propriétés sont des cas particuliers des suivantes.

En tenant compte des résultats obtenus (Section XXI), sur les congruences du triangle arithmétique, la formule (152) donne, pour p premier, et $n = p - 1$,

(154) $$E_{p-1} + E_{p-3} + E_{p-5} + \ldots + E_2 + E_0 \equiv 0, \quad (\text{Mod. } p);$$

on a donc cette proposition:

* *Journal de Crelle*, t. 79, pag. 67.

THÉORÈME: *Si p est un nombre premier, la somme des nombres Eulériens, pris avec les signes alternés + et —, dont l'indice est plus petit que p, est divisible par p.*

Les premières valeurs de E sont donnés par les relations

$$\begin{aligned} E_2 + E_0 &= 0, \\ E_4 + 6E_2 + E_0 &= 0, \\ E_6 + 15E_4 + 15E_2 + E_0 &= 0, \\ \ldots\ldots\ldots\ldots &, \\ \ldots\ldots\ldots\ldots &; \end{aligned}$$

on a ensuite, par congruence, à partir de E_{p+1}, pour le module premier p

$$\begin{aligned} E_{p+1} + E_0 &\equiv 0, \\ E_{p+3} + 3E_{p+1} + 3E_2 + E_0 &\equiv 0, \\ E_{p+5} + 10E_{p+3} + 5E_{p+1} + 5E_4 + 10E_2 + E_0 &\equiv 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots &, \\ \ldots\ldots\ldots\ldots\ldots\ldots &; \end{aligned}$$

la comparaison des deux groupes de formules qui précédent, donne successivement

$$(155) \qquad \begin{aligned} E_{p+1} &\equiv E_2, \quad (\text{Mod. } p), \\ E_{p+3} &\equiv E_4, \quad \text{“} \\ E_{p+5} &\equiv E_6, \quad \text{“} \\ \ldots\ldots & \end{aligned}$$

On obtient de même, en général. pour k entier quelconque

$$(156) \qquad E_{2n+k(p-1)} \equiv E_{2n}, \quad (\text{Mod. } p),$$

et, par suite:

THÉORÈME: *Les résidus des nombres Eulériens, suivant un module premier quelconque, se reproduisent périodiquement dans le même ordre, comme les résidus des puissances des nombres entiers.*

Posons maintenant

$$u^a = \left(\frac{2}{e^x + e^{-x}}\right)^a = E_{a,0} + E_{a,1}\frac{x}{1} + E_{a,2}\frac{x^2}{1.2} + \ldots + E_{a,n}\frac{x^n}{1.2\ldots n} + \ldots$$

ou, sous la forme symbolique

$$(157) \qquad u^a = e^{E_a x},$$

les coefficients $E_{a,n}$ sont déterminés par la relation symbolique

$$(158) \qquad E_{a,n} = [E' + E'' + E''' + \ldots + E^{(a)}]^n,$$

dans le développement de laquelle on remplace les exposants de $E', E'', \ldots E^{(a)}$ par des indices, et en supprimant les accents. On a aussi la formule

(159) $$E_{a,n} = [E_{a-1} + E_1]^n,$$

dans laquelle en remplace les exposants de E_{a-1} et de E, par des seconds indices. Ces nombres $E_{a,n}$ que nous appellerons les *nombres Eulériens d'ordre* a sont entiers pour a entier et positif; on démontre, comme ci-dessus, que leurs résidus suivant un module premier se reproduisent périodiquement, et que l'on a encore

(160) $$E_{a,n} \equiv E_{a,n+k(p-1)}, \quad (\text{Mod. } p).$$

Ces considérations s'appliquent, en général, aux coefficients différentiels d'une fraction rationnelle de e^x, mais, dans certaines conditions, comme dans le cas de

$$\frac{\phi(1)}{\phi(e^x)}.$$

Cependant, lorsque ϕ (1) est nul, comme dans le développement de $\frac{1}{1-e^x}$ qui contient les nombres de BERNOULLI, ce théorème ne se présente plus immédiatement, puisque les coefficients ne sont plus entiers, et contiennent en dénominateur une série indéfinie de nombres premiers. Alors, on les multiplie par d'autres fonctions telles que

$$a(a^n - 1)$$

afin de les rendre entiers, et d'appliquer les résultats qui proviennent des congruences du triangle arithmétique.

PARIS, Décembre, 1877.

ON THE TWO GENERAL RECIPROCAL METHODS IN GRAPHICAL STATICS.

By Henry T. Eddy, *Cincinnati, Ohio.*

§ 1.

The methods employed in the geometrical or graphical solution of statical problems are, in fact, merely applications of the parallelogram of forces, so systematized and combined that the skilful draughtsman is able, by these geometrical processes alone, to make computations sufficiently exact for practical purposes with a rapidity and insight into the real relations of the quantities treated which often far surpasses that of any algebraic or numerical process.

These geometrical processes are of two principal kinds. The first kind determines the stress in each piece of any given framed structure, in which the stresses are determinate, by drawing a reticulated polygon whose lines represent these stresses and the applied forces. If a certain order of procedure be observed in drawing this reticulated force polygon, the frame and force polygon stand in a reciprocal relationship to each other, which has been clearly set forth elsewhere.* The second kind of geometrical process employed aims at somewhat more general relations than those obtained by the force polygon, and applies not only to any framed structure considered as a single elastic piece of material, but to any elastic piece, framed or not. The greater generality attained by processes of the latter kind is due to the assumption of such arbitrary forms of framing that their geometrical properties are of material assistance in determining the magnitudes sought.

Hitherto, one process only has been known which possesses this characteristic generality, which process is based upon the properties of the *equilibrium polygon.* An equilibrium polygon is often called a catenary or funicular polygon, since it is the form assumed by a perfectly flexible string under the action of the forces applied to it. Some of its geometrical properties have

* Reciprocal Figures. James Clerk Maxwell. Philosophical Magazine, vol. 27. London, 1864.
Le figure reciproche nelle statica grafica. Luigi Cremona. Milano, 1872.
A New General Method in Graphical Statics. Henry T. Eddy. Van Nostrand's Engineering Magazine, vol. 18. New York, 1878.

Plate IV.

EQUILIBRIUM POLYGON METHOD.

Fig. 1.

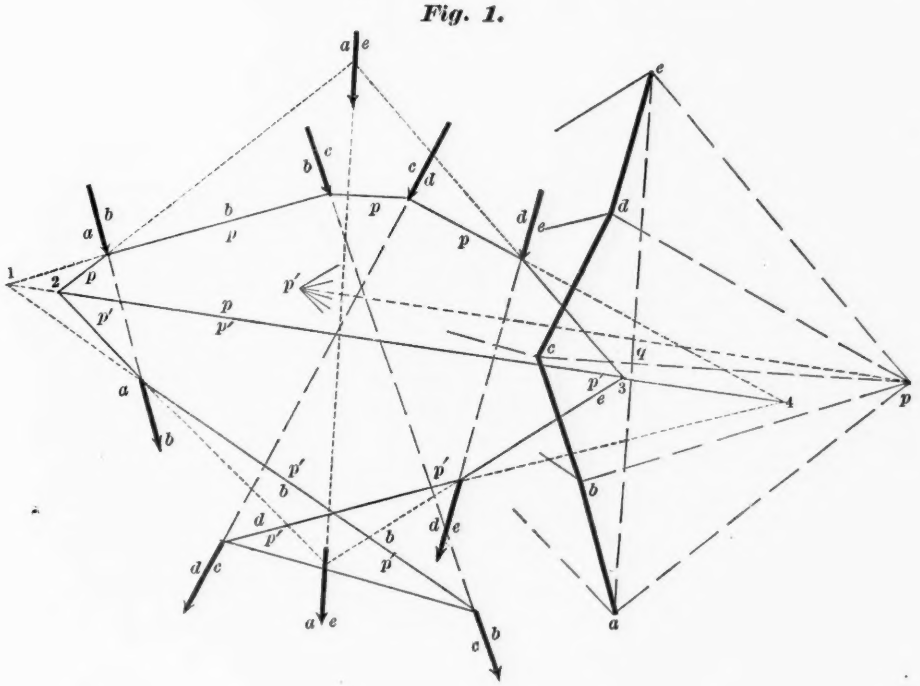
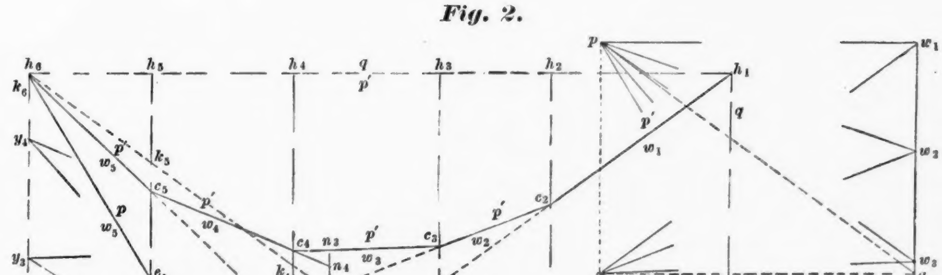


Fig. 2.



EDDY, *On the Two General Reciprocal Methods in Graphical Statics.*

Plate V.

FRAME PENCIL METHOD.

Fig. 3.

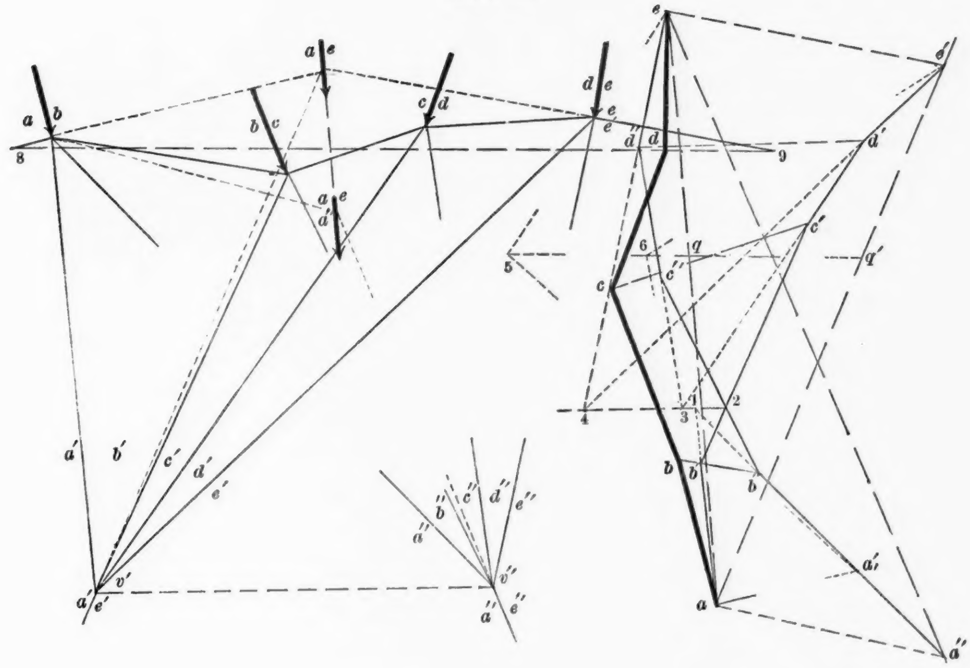
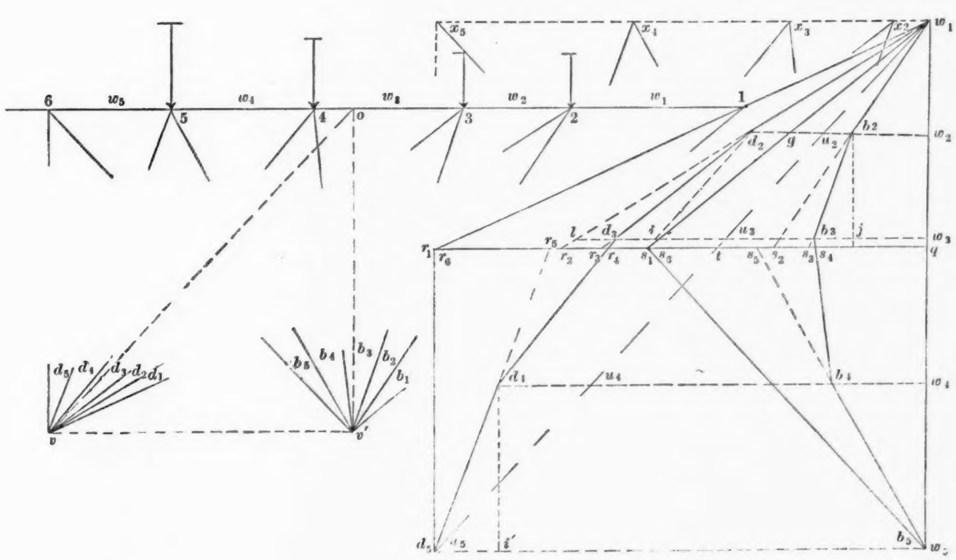


Fig. 4.



EDDY, *On the Two General Reciprocal Methods in Graphical Statics.*

been long known, but the systematic development and practical application of the geometrical relations involved in the equilibrium polygon and its reciprocal force polygon is comparatively recent,* and has been principally dependent upon the growth of modern higher geometry.

Culmann is justly regarded as the father of graphical statics as a practical method, and to him is due the establishment of the generality and importance of the equilibrium polygon method. Mohr discovered an important extension of the method † in showing that the deflection curve of a straight elastic girder is a second equilibrium polygon, and the author has further extended the method by showing its applicability to any curved elastic girder or arch. ‡ Many other writers have added to the subject and simplified it. The bibliography of the subject will be found in sufficient detail elsewhere. §

In a paper by Poncelet, || as given with some modifications by Woodbury, ¶ the germs of a second fundamental method appear, which is of the same general nature as that of the equilibrium polygon to which it bears a certain kind of reciprocal relationship; but neither the author nor any subsequent writer seems to have seen the possibility of establishing a general graphical method of which this special solution would be a particular case.

The purpose of this paper is to establish the general properties of this new general method, and to point out the reciprocity existing between it and that of the equilibrium polygon. In attempting this, it seems best to establish the general properties of the equilibrium polygon from mechanical considerations, instead of deriving them from higher geometry, and then to obtain the corresponding properties of the new method, which we have ventured to call the *frame pencil* method for reasons which will appear subsequently.

§ 2.

In Fig. 1, let the forces which it is proposed to treat lie in one plane and have their lines of action along the arbitrarily assumed lines, on each side of

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* Graphische Statik. C. Culmann. Zurich, 1866. Also, vol. I, 2d ed. Zurich, 1875.

† Beitrag zur Theorie der Holz- und Eisenconstructionen. Zeitschr. d. Hannov. Ing. und Arch. Ver. 1868.

‡ New Constructions in Graphical Statics. Henry T. Eddy. Van Nostrand's Engineering Magazine. Vol. 16. New York, 1877.

§ Ueber die Graphische Statik. J. I. Weyrauch. Leipzig, 1874. Translated as an Introduction to The Elements of Graphical Statics. A. J. Du Bois. New York, 1875.

Lezioni di Statica Grafica. Antonio Favaro. Padova. Of this work there is a forthcoming French translation. Gauthier-Villars. Paris.

|| Mémorial de l'officier du génie. No 12.

¶ Stability of the Arch. D. P. Woodbury. New York, 1858.

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which are the letters *ab*, *bc*, *cd*, *de* respectively: these lines may be called a *diagram* of the forces, and this kind of notation is employed for convenience of clearly showing the reciprocity between the diagram occupying the left of the figure and the force polygon on the right, which we shall now construct. Draw the lines at the extremities of which are *ab*, *bc*, *cd*, *de*, in such directions and of such lengths that they shall severally represent, on some convenient scale, the magnitudes as well as the directions of the forces. This will constitute the *polygon* of the applied forces. Since the force diagram shows on some convenient scale of distances the relative position and direction of the forces, and the force polygon shows their relative magnitude and also their directions, it is evident that *ab*, etc., of the diagram is parallel to *ab*, etc., of the polygon, and the number or position of the forces which can be treated is in no way restricted to those we have arbitrarily chosen to illustrate the method It is necessary, in drawing the force polygon *abcde*, to have regard to the signs of the forces, so that, in passing continuously along the polygon, the motion should be all in the direction in which the forces act, and not partly with and partly against the forces. The arrow-heads show the sense in which each force is taken, and the polygon is taken in the same sense by passing continuously from *e* to *a*.

Next, assume any pole *p* as the common point of the rays *pa*, *pb*, etc., of the force pencil *p – abcde*. The lengths of the rays *pa*, etc., represent, on the assumed scale of forces, the magnitudes of the stresses in the members of a frame of which we shall immediately draw a diagram; and the directions of *pa*, etc., show what directions the members of the frame must have. On the left, draw the line lying between the letters *pa* parallel to the ray *pa*:— this line, as will be seen later, is one side of an equilibrium polygon, and so it will be called the *side pa* to distinguish it from the *ray pa*. The actual position of the side is of no consequence; its parallelism to the ray is the only important consideration. From the point at which the side *pa* intersects the diagram of the force *ab*, draw the side *pb* parallel to the ray *pb*: and from the intersection of the side *pb* with the diagram of *bc* draw the side *pc* parallel to the ray *pc*. In the same manner draw a side parallel to each of the rays, so that finally the polygon *p – abcde* has a side parallel to each of the rays of the pencil *p – abcde*; and the two are said to be reciprocal figures. The reciprocity is that usually existing between a frame and its force polygon, and it is that pointed out in connection with the kind of graphical process first mentioned.

Consider the forces acting at the point *pab* of the diagram, supposing that two members *pa* and *pb* of a frame meet here in a perfectly flexible hinge joint and hold the force *ab* in equilibrium. From the parallelogram of forces it is known that the sides of the triangle *pab* represent the relative magnitudes of the forces in equilibrium at the joint *pab*. Similarly, the sides of the triangle *pbc* represent the relative magnitudes of the forces held in equilibrium at the hinge joint *pbc;* and in the same manner the forces meeting at the successive joints are represented in relative magnitude by the sides of the triangle denoted by the same letters.

The sides *pa*, *pb*, etc., together, form an equilibrium polygon, so called because such a frame requires no internal bracing in order to sustain the applied forces. In the case we have taken, it is evident that the stresses are all compressive, and the frame would form an equilibrated arch.

In the system of notation here used, *p* is regarded as denoting, on the left, the area enclosed by the equilibrium polygon, *b* as the area between the lines *ab*, *pb* and *cb*, and *c* as that between *bc*, *pc* and *dc*, whether these converge or diverge, *i. e.*, whether the space so bounded is finite or not.

Close the force polygon by drawing the side *ae*, then *ae* represents the relative magnitude of the resultant of the applied forces, or the force which will hold them in equilibrium according to the sense in which it is taken; for forces proportional to the sides of a closed polygon have no resultant. The side *ae*, also, is in the direction of the resultant, the only remaining question being as to its position. Prolong the first side *pa* until it intersects the last side *pe*, and through this intersection draw the diagram of a force *ae* parallel to the closing side *ae* of the force polygon. The diagram *ae* gives the true position of the line of application of the resultant; for, suppose the prolongations of *pa* and *pe* to be members of the frame with a hinge joint at their intersection, then the sides of the triangle *pae* represent the relative magnitudes of the forces acting at this joint, when a force is applied at this point which will hold the other applied forces in equilibrium.

Draw any line intersecting the sides *pa* and *pe* as 2 3, and also *pq* in the force polygon, parallel to 2 3; then may the points 2 and 3 be taken as fixed points of support of the equilibrium polygon or arch to which the forces are applied. This arch has the span 2 3 and the applied forces cause a thrust along 2 3, whose magnitude is given by the length of *pq*. This thrust along *pq* may be sustained by a member joining 2 3 or by these joints in virtue of

their being fixed. They, in either case, together sustain the resultant which is divided so that 2 and 3 sustain aq and qe respectively, as clearly appears from the fact that the triangles paq and qep represent the forces in equilibrium at 2 and 3 respectively. The line 2 3 is called the *closing line* of the polygon p.

Again, choose any pole p' as the common point of the force pencil $p' - abcde$. To avoid multiplying lines p' has been taken upon pq prolonged. Draw the equilibrium polygon p' whose sides are parallel to the rays of the pencil $p' - abcde$, and to still further avoid the multiplication of lines, let the side $p'a$ pass through the point 2. The sides of the polygon p' are all under tension except $p'c$, but they might all have been either in tension or all in compression had the side $p'a$ been made to pass through some point other than 2.

As shown in connection with the polygon p the first and last sides of any polygon must intersect on the diagram of the resultant, hence $p'a$ and $p'e$ intersect on the line ae already drawn.

Prolong the corresponding sides pa and $p'a$, pb and $p'b$, etc., until they intersect at 2, 1, etc., then are the points 1 2 3, etc., upon one and the same straight line. For, suppose the forces which are applied to the polygon p' to be reversed in direction, then the system applied to the polygons p and p' must together be in equilibrium; and the only bracing needed is the common member 2 3 parallel to pp', since the forces applied to p produce a thrust pq along it, and those applied to p' a thrust qp', while the reactions aq and qe at 2 and 3 are in equilibrium. A similar result holds for each of the forces separately; *e. g.* the opposed forces ab acting on p and p' may be considered to be applied at opposite points of a quadrilateral whose remaining joints are 1 and 2; the force polygon corresponding to this quadrilateral is $apbp'$; hence 1 2 is parallel to pp'. But 2 3 is parallel to pp', therefore 1 2 3 are in one and the same straight line. The same may be shown respecting the remaining intersections of corresponding sides. The intersection of pc and $p'c$ does not fall within the limits of the figure.

The properties which have been established with regard to the relations of force diagram and equilibrium polygon to the force polygon and force pencil are really of a geometric nature, and are not dependent upon the fact that they represent relationships between forces. The proposition may be stated in geometric language thus: from any points $abcd$, etc., draw lines converging to a single point as pole, also from the same number of points

1 2 3 4, etc., lying in a straight line; draw lines so that there shall be one line from each of the points 1 2 3 4, etc., parallel to each different convergent through *abcd*, etc.; then, if the common point of the convergent lines be removed parallel to the line 1 2 3 4, so that the convergents severally revolve about the points *abcd*, etc., and the lines from 1 2 3 4 which are respectively parallel to the convergents also revolve severally about 1 2 3 4, etc., the loci of any and all of the intersections of these last mentioned lines are straight lines, which are parallel to the lines joining the points *ab*, *bc*, *ac*, etc.; and conversely, if these loci are the last mentioned straight lines, they revolve about the fixed points 1 2 3 4, etc., which lie in one and the same straight line.

For convenience of reference, we have here collected the special terms by which the various parts of the reciprocal figures in the left and right are designated.

In Direction and Position.	Force Diagram,	*abcde*,	Force Polygon.	In Direction and Magnitude.
	Equilibrium Polygon,	$p - abcde$,	Force Pencil.	
	Closing Line,	$2\ 3 \parallel pq$,	Resolving Ray.	
	Diagram of Resultant,	*ae*,	Resultant Force.	

§ 3.

Most of the useful applications of graphical methods treat some system of parallel forces, in which case, the equilibrium polygon has additional properties of importance which will now be exhibited.

Let the system of parallel forces be that represented in Fig. 2, viz: let the verticals 2 3 4, etc., between $w_1\,w_2\,w_3$, etc., be the diagrams of the applied forces, of which the relative magnitudes are w_1w_2, w_2w_3, etc., in the force polygon on the right. The force polygon in this case becomes a straight line (often called the *weight line*) and the closing side $w_5\,w_1$ of the force polygon is in the same straight line with the other sides $w_1\,w_2$, etc.

Assume any pole p of the force pencil $p - ww$, and construct the equilibrium polygon $p - ww$ or *ee*, whose sides are parallel to the rays of the force pencil, in the manner which has been previously explained. Draw the closing line *kk* of the polygon *ee* through the points k_1 and h_6 where the first and last sides of the equilibrium polygon intersect the verticals 1 and 6, which last are assumed to be the lines of support for the applied forces. Draw the closing ray *pq* parallel to *kk*; then, as was before shown, it divides the resultant

force $w_1\,w_5$ at q into the two parts which rest on the supports in the verticals 1 and 6. The diagram of the resultant is in the vertical mm through the intersection of the first and last sides of ee, as was also previously shown.

Choose a second pole p' from which to draw a force pencil $p'-ww$. Since this pole p' has been taken on a horizontal through q, the new closing line hh of the equilibrium polygon cc, whose sides are parallel to the rays of this pencil, will then be also horizontal. The first and last side will intersect on the vertical mm before found; and corresponding sides and diagonals of the polygons ee and cc all intersect in one and the same straight line yy, which is parallel to pp', as was previously proven. The coincidences just mentioned would, in any practical case, afford a most complete series of checks and tests of accuracy in drawing.

The line pp' and its parallel yy have, in this figure, been made vertical, so that p and p' are equidistant from ww. Designate the horizontal distance from p or p' to the weight line ww by the letter H. It happens in Fig. 2 that $pw_1 = H$, but in any case the pole distance H is the horizontal component of the force pq acting along the closing line kk.

Now by similarity of triangles,

$$k_1e_2\ (= h_1h_2) : k_2e_2 :: pw_1 : qw_1 \qquad \therefore\ H.\,k_2e_2 = qw_1\,.\,h_1h_2 = M_2\,,$$

the moment of flexure, or bending moment, at the vertical 2, which would be caused in a simple straight beam or girder sustaining the given weights w_1w_2, etc., and resting upon supports in the verticals 1 and 6.

Again, from similarity of triangles,

$$k_1f_3\ (= h_1h_3) : k_3f_3 :: H : qw_1$$
$$e_2f_3\ (= h_2h_3) : e_3f_3 :: H : w_1w_2$$
$$\therefore\ H\,(k_3f_3 - e_3f_3) = H.\,k_3e_3 = qw_1\,.\,h_1h_3 - w_1w_2\,.\,h_2h_3 = M_3\,,$$

the moment of flexure of the simple girder at the vertical 3.

Similarly, it can be shown in general that

$$H.\,ke = M,$$

i. e. that the moment of flexure at any vertical whatever (be it one of the verticals 2 3 4, etc., or not) is the product of the assumed pole distance H and the vertical ordinate ke included between the equilibrium polygon ee and its closing line kk.

Evidently the same properties can be shown to hold respecting the vertical ordinates of the polygon cc, from which it is seen that $k_2e_2 = h_2c_2$, etc., and $H.\,ke = H.\,hc = M$.

From the foregoing it appears that the equilibrium polygon for parallel forces is a moment curve, *i. e.*, its vertical ordinate at any point of the span is proportional to the bending moment at that point in a girder sustaining the given weights, and supported by resting without constraint upon piers at its extremities.

From this demonstration it is clear that

$$H\,.\,e_3f_3 = w_1w_2\,.\,h_2h_3\,, \quad H\,.\,m_1m_2 = w_1w_2\,.\,e_2m_1\,, \quad H\,.\,y_1y_2 = w_1w_2\,.\,h_2h_6$$

are respectively the moments of the force w_1w_2 about the vertical 3, about the vertical *mm* through the center of gravity, and about the vertical 6.

Similarly, m_1m_3 is proportional to the moment of all the forces at the right, and m_3m_5 to all at the left of the center of gravity, but $m_1m_3 + m_3m_5 = 0$, as should be the case at the center of gravity, about which the moments of the applied forces vanish.

From these considerations, it appears that the segments *mm* or *nn* of the resultant are proportional to the bending moments caused by the weights at the center of gravity of a girder sustaining the given weights and resting, without constraint, upon a single support at their center of gravity.

Also, the segments y_1y_2, y_2y_3, etc., are proportional to the bending moments caused at the vertical 6 by the weights w_1w_2, w_2w_3, etc., at the vertical 6, in a girder which sustains these weights, in case it is firmly fixed and built in at this vertical, and has no other support.

A vertical to which the resultant force is transferred, as it is in this case to *yy*, by aid of a couple introduced at that vertical, may be conveniently designated as a pseudo-resultant. The magnitude of the moment of the couple here introduced is $H\,.\,y_1h_6 = y_1m_1\,.\,w_1w_5$, and by it the closing line *kk* is often said to be moved to the position k_1y_1, but it seems preferable to call this last a pseudo-closing line due to the kind of constraint and support at the vertical 6.

§ 4.

In Fig. 3, let *ab*, *bc*, *cd*, *de* be the diagrams of the system of applied forces, and *abcde* the corresponding force polygon; choose one point arbitrarily upon the line of action of each of these forces, and join these points to any assumed vertex v' by rays of the *frame pencil* $a'b'c'd'e'$. Also, join successively the points chosen by the lines bb', cc', dd', which form sides of what may be called the *frame polygon.*

Now, consider the given forces to be sustained by the frame pencil and frame polygon as a system of bracing, which system exerts a force at the vertex v' in some direction not yet known, and also exerts a force along some arbitrarily assumed member ee' which may be regarded as forming a part of the frame polygon. From the points $bcde$ in the force polygon, draw the *force lines* bb', cc', dd', ee' parallel to the sides bb', etc., of the frame polygon, and commencing at a, draw the *equilibrating force polygon* $ab'c'd'e'$, whose sides are parallel to the successive rays of the frame pencil $a'b'c'd'e'$: then the stresses upon the rays of the frame pencil will be given in relative magnitude by the lengths of the corresponding sides of the equilibrating polygon, and the stresses upon the sides bb', etc., of the frame polygon by the lengths of the force lines bb', etc. These statements are shown to hold true from the fact that, in the right hand figure, in which lengths of lines represent magnitudes of forces, a closed polygon will be found whose sides are parallel to the directions of the forces meeting at each joint of the frame. The notation, as previously employed, will assist in the ready identification of the corresponding parts of the figures on the right and left.

If a *resultant side* ae' be drawn in the equilibrating polygon, and also parallel to it from v', a *resultant ray* $a'e'$, then the ray $a'e'$ will intersect the side ee' at a point in line of action of the resultant of the given system of forces; for this intersection is such a point that if the resultant alone be applied at it, there will be the same stresses along the members $a'e'$ and ee' as the applied forces themselves produce in those members, as appears when we consider that the triangle aee' represents the forces in equilibrium at the point in question. The diagram of the resultant is ae, parallel to the side ae of the force polygon.

If the arbitrarily assumed member ee' be revolved about its intersection with de, then the force line ee' will revolve about e, and e' will move along the side $d'e'$, and it appears that the locus of the intersection of the corresponding positions of the resultant ray $a'e'$ and the last side ee' will be the diagram of the resultant ae.

A first side aa' of the frame polygon can now be drawn corresponding to the assumed last side ee', and its significance can be readily seen as follows: draw the force line aa'_1 parallel to this side aa'; then the equilibrating polygon $ab'c'd'e'$, the first side of which passes through a, can be begun at any point of aa'_1, and the closing will be parallel to its present position (as is obvious from mechanical considerations), but the stresses in each of the members of

the frame will have been changed by this supposition. The case previously assumed was that in which the stress in the side aa' is zero.

It is usually a most convenient practical simplification to make all the sides of the frame polygon lie in one and the same straight line, which may be called the *frame line;* then, since the force lines are all parallel to it, the direction of the line aa'_1 is known at once, (it being one of these parallels), and the stresses in the rays of the frame pencil are the same, whatever be the point of aa'_1 at which the equilibrating polygon is begun.

It should be noticed that the equilibrium polygon is also one case of the frame polygon.

Suppose the points 8 and 9 of the first and last sides respectively to become fixed points of support, the thrust between these points may be sustained by a member 8 9, or by 8 and 9 regarded as abutments. To find the point q of the resultant force ae, at which it is divided into the two parts sustained at 8 and 9, draw $a6$ parallel to $v'8$, and $e'6$ parallel to $e'9$, and then through 6 draw the *resolving line* qq' parallel to 8 9.

This may be regarded as the same geometric proposition as that previously proven, in which it was shown that the locus of the intersection of the first and last sides of an equilibrium polygon (reciprocal to a given force pencil) is parallel to the resultant side of the force polygon, and is the diagram of the resultant. The proposition now is that the locus of the equilibrating polygon (reciprocal to a given frame pencil) is parallel to the closing side 8 9 of the frame polygon and is the resolving line. These two statements are geometrically equivalent.

Assume a second vertex v'', and draw the frame pencil and its corresponding equilibrating polygon $ed''c''b''a''$. The last point of the equilibrating polygon is at a'' or at a'_1 according as aa'' or aa' be taken as the arbitrary side of the frame polygon.

The intersection of the resulting ray $a''e''$, parallel to the side $a''e$, with the arbitrary side aa'' is on the diagram ae of the resultant, as has been shown. Also, if aa' be taken as the arbitrary side, it has been shown that $a'_1 5$ and $e5$, respectively parallel to $v''8$ and $v''9$, intersect upon the resolving line qq'.

Again, the corresponding sides of these two equilibrating polygons intersect at 1 2 3 4, points which are upon one and the same straight line parallel to $v'v''$; for this is the same proposition respecting two vertices and two equilibrating polygons which was previously proved respecting two poles and two equilibrium polygons.

For convenient reference, a table of the special names given to the different parts of the reciprocal figures just treated is here inserted:

In Direction and Position.	Force Diagram,	$abcde$,	Force Polygon.	In Direction and Magnitude.
	Frame Pencil,	$a'b'c'd'e'$,	Equilibrating Polygon.	
	Frame Polygon,	bb', cc', dd', ee',	Force Lines.	
	Resultant Ray,	$a'e'$,	Resultant Side.	
	Closing Side,	$8\,9 \parallel qq'$,	Resolving Line.	
	Diagram of Resultant,	ae,	Resultant Force.	

§ 5.

Let the same system of parallel forces, which was treated by the equilibrium polygon method in Fig. 2, be also treated by the frame pencil method in Fig. 4. Suppose them to be applied to a horizontal girder at 2 3 4 5, and let it be supported at 1 and 6.

Use 1 6 as the frame line, and choose any vertex v, arbitrarily, from which to draw the rays of the frame pencil dd. As has been previously shown, if a resultant ray vo of the frame pencil dd be drawn from v parallel to the closing line uu of the equilibrating polygon dd, this ray intersects 1 6 at the point o, at which the diagram of the resultant intersects 1 6.

Furthermore, the lines w_1r_1 and d_5r_6, respectively parallel to the abutment rays $v1$ and $v6$ of the frame pencil, intersect on the resolving line rr, which determines the point of division q of the reactions of the supports, as was before shown.

Let the vertical distance of the vertex v from the frame line 1 6 be denoted by the letter V. It happens in Fig. 4 that $v6 = V$. When, however, the frame polygon is not straight, or, being straight, is inclined to the horizon, V has different values at the different joints of the frame polygon; but in every case V is the vertical distance of the joint under consideration above or below the vertex. This possible variation of V is found to be of practical use in certain constructions.

By similarity of triangles we have

$$1\,2 : v6 :: r_1r_2 : w_1q \quad \therefore\ V \,.\, r_1r_2 = w_1q \,.\, 1\,2 = M_2\,,$$

the bending moment of girder at 2.

Draw a line through w_1 parallel to $v3$, this line by chance coincides so nearly with w_1s_1 that we will consider that it is the line required, though it was drawn for another purpose.

Again, by similarity of triangles

$$1\,3 : v6 :: r_1s_1 : w_1q, \qquad 2\,3 : v6 :: d_2g\ (= r_3s_1) : w_1w_2$$

$$\therefore\ V\,(r_1s_1 - r_3s_1) = V.\,r_1r_3 = w_1q\,.\,1\,3 - w_1w_2\,.\,2\,3 = M_3$$

the bending moment at 3.

Similarly, it may be shown that

$$V.\,r_1r_n = M_n\,,$$

i. e., that the moment of flexure at any point of application of a force to the girder is the product of the assumed vertical distance V multiplied by the corresponding segment rr of the resolving line.

To find the moment of flexure at *any* point of the girder, draw a line tangent to the equilibrating polygon (or curve) and parallel to a ray of the frame pencil at that point, then the intercept r_1r of this tangent is such that $V.\,r_1r$ is the moment required.

Also, by similarity of triangles,

$$o2 : v6 :: u_2d_2 : w_1w_2\,, \qquad \therefore\ V.\,u_2d_2 = w_1w_2\,.\,o2$$

$$o2\ (= o3 + 3\,2) : v6 :: u_3l : w_1w_3\,, \qquad 3\,2 : v6 :: d_3l : w_2w_3$$

$$\therefore\ V\,(u_3l - d_3l) = V.\,u_3d_3 = w_1w_2\,.\,o2 + w_2w_3\,.\,o3\,,$$

i. e., the horizontal abscissas ud between the equilibrating polygon dd and its closing side uu multiplied by the vertical distance V, are the algebraic sum of the moments of the forces about their center of gravity. The moment of any single force about the center of gravity being the difference between two successive algebraic sums may be found thus: draw d_2i parallel to uu, then is $V.\,d_3i$ the moment of w_1w_2 about the center of gravity, as may be also proved by similarity of triangles.

Again, by proportions derived from similar triangles precisely like those already employed, it appears that

$$V.\,w_2d_2 = w_1w_2\,.\,2\,6$$

is the moment of the force w_1w_2 about the point 6. And similarly it may be shown that

$$V.\,w_3d_3 = w_1w_2\,.\,2\,6 + w_2w_3\,.\,3\,6$$

is the moment of w_1w_2 and w_2w_3 about 6.

Furthermore, as this point 6 was not specially related to the points of application 1 2 3 4, we have thus proved the following property of the equilibrating polygon: if $v6$, a pseudo-resultant ray of the frame pencil, be drawn to any point of the frame line, then the horizontal abscissas between the equilibrating polygon and a side ww parallel to that ray (which may be called a pseudo-closing side), are proportional to the sum total of the moments,

about that point, of those forces which are found between that abscissa and the end of the weight line from which this pseudo side was drawn. The difference between two successive sum totals being the moment of a single force, a parallel to the pseudo side determines at once the moment of any force about the point; *e. g.* draw d_4i' parallel to ww, therefore $V.d_5i'$ is the moment of w_4w_5 about 6.

Now move the vertex to a new position v' in the same vertical with o: this will cause the closing side of the equilibrating polygon (parallel to $v'o$) to coincide with the weight line. The new equilibrating polygon bb has its sides parallel to the rays of the frame pencil whose vertex is at v'. If V is unchanged, the abscissas and segments of the resolving line are unchanged, and vv' is horizontal. Also xx parallel to vv' contains the intersections of corresponding sides and diagonals of the equilibrating polygon. These statements are geometrically equivalent to those made and proved in connection with the equilibrium polygon and force pencil.

§ 6.

In Figs. 2 and 4 we have taken $H = V$, hence the following equalities will be found to exist:

$$k_2e_2 = r_1r_2\,,\quad k_3e_3 = r_1r_3\,,\quad k_4e_4 = r_1r_4\,,\ \text{etc.}$$
$$m_1m_2 = u_2d_2\,,\quad m_1m_3 = u_3d_3\,,\quad m_1m_4 = u_4d_4\,,\ \text{etc.}$$
$$y_1y_2 = w_2d_2\,,\quad y_1y_3 = w_3d_3\,,\quad y_1y_4 = w_4d_4\,,\ \text{etc.}$$
$$m_2m_3 = d_3i\,,\ \text{etc.}\quad y_4k_6 = d_5i'\,,\ \text{etc.}$$

By the use of *etc.*, we refer to the more general case of many applied forces as well as to the remaining like equalities in Figs. 2 and 4.

From these equations the nature of the relationship existing between the method of the equilibrium polygon and its force pencil and the method of the frame pencil and its equilibrating polygon becomes clear. It may be stated in words as follows:

The height of the vertex (a vertical distance) and the pole distance (a horizontal force) stand as the type of the reciprocity or correspondence to be found between the various parts of the figures. The ordinates of the equilibrium polygon (vertical distances) correspond to the segments of the resolving line (horizontal forces), each of these being proportional to the bending moments of a simple girder sustaining the given weights, and resting without constraint upon supports at its two extremities.

The segments of the resultant line (vertical distances) correspond to the abscissas of the equilibrating polygon, (horizontal forces), each of these being proportional to the bending moments of a simple girder sustaining the given weights and resting without constraint upon a support at their center of gravity.

The segments of any pseudo resultant line parallel to the resultant which are cut off by the sides of the equilibrium polygon, are proportional to the bending moments of a girder supporting the given weights and rigidly built in and supported at the point where the line intersects the girder: to these segments correspond the abscissas between the equilibrating polygon and a pseudo side of it parallel to the pseudo resultant ray.

The two different kinds of support which we have supposed, viz: support without constraint and support with constraint, can be treated in a somewhat more general manner, as appears when we consider that at any point of support there may be besides the reaction of the support a bending moment, such as would be induced, for instance, when the span in question forms part of a continuous girder, or when it is fixed at the support in a particular direction. In such a case the closing line of the equilibrium polygon is said to be moved to a new position. It seems better to call it in its new position a pseudo closing line. The ordinates between the pseudo closing line and the equilibrium polygon are proportional to the bending moments of the girder so supported. It is possible to induce such a moment at one point of support as to entirely remove the weight from the other, and cause it to exert no reaction whatever; and any intermediate case may occur in which the total weight in the span is divided between the supports in any manner whatever. When the weight is entirely supported at h_6 of Fig. 2 then y_1e_2 is the pseudo-closing line of the polygon *ee*. In that case xx of Fig. 4 becomes the pseudo-resolving line, and in general the ordinates between the pseudo-closing line and the equilibrium polygon correspond to the segments of the pseudo-resolving line, and are proportional to the bending moments of the girder. This general case is not represented in Figs. 2 and 4;* but the particular case shown, in which the total weight is borne by the left pier, gives the equations

$$e_3f_3 = w_1x_2\,, \quad e_4f_4 = w_1x_3\,, \quad e_5f_5 = w_1x_4\,, \text{ etc.}$$

* See page 82, Researches in Graphical Statics. Henry T. Eddy. New York, 1878.

DEMONSTRATION OF A FUNDAMENTAL THEOREM OBTAINED BY MR. SYLVESTER.

BY B. LIPSCHITZ, *Professor Ordinarius in the University of Bonn.*

IN a memoir published in the 85th volume of Mr. Borchart's Journal, Mr. Sylvester has developed the thought of a new notation for algebraical forms or quantics. Being given any algebraical homogeneous function of the n variables $x_1, x_2, \ldots x_n$, whose degree is denoted by p, let the powers and products of powers of the said degree $x_1^p, x_1^{p-1}x_2 \ldots x_n^p$ be expressed by $X_1, X_2, \ldots X_\nu$, and the corresponding polynomial coefficients by $\pi_1, \pi_2, \ldots \pi_\nu$, then the proposed form may be expressed by applying as numerical factors to the constants $f_1, f_2, \ldots f_\nu$ the square roots of the respective polynomial coefficients, so that

$$(1) \qquad F(x_1, x_2, \ldots x_n) = \sqrt{\pi_1} f_1 X_1 + \sqrt{\pi_2} f_2 X_2 + \ldots + \sqrt{\pi_\nu} f_\nu X_\nu,$$

and is called in that shape a *prepared* form. Now suppose that on introducing instead of the n variables $x_1, x_2, \ldots x_n$ the n linear functions of n new variables $y_1, y_2, \ldots y_n$, such that

$$(2) \qquad x_a = k_{a,1} y_1 + k_{a,2} y_2 + \ldots + k_{a,n} y_n,$$

where the letter a runs through the numbers $1, 2, \ldots n$, the form $F(x_1, x_2, \ldots x_n)$ is changed into the form $G(y_1, y_2, \ldots y_n)$, written likewise as a prepared form, so that

$$(3) \qquad G(y_1, y_2, \ldots y_n) = \sqrt{\pi_1} g_1 Y_1 + \sqrt{\pi_2} g_2 Y_2 + \ldots + \sqrt{\pi_\nu} g_\nu Y_\nu .$$

It is always understood that in substituting for the letter x the letters y, z, t, u, the functions X_a are respectively turned into the functions Y_a, Z_a, T_a, U_a, the letter a going through the numbers $1, 2, \ldots \nu$. As the constant elements $g_1, g_2, \ldots g_\nu$ depend in a linear manner upon the elements $f_1, f_2, \ldots f_\nu$, we get by partial differentiation the ν equations

$$(4) \qquad \begin{aligned} g_1 &= \frac{\delta g_1}{\delta f_1} f_1 + \frac{\delta g_1}{\delta f_2} f_2 + \ldots + \frac{\delta g_1}{\delta f_\nu} f_\nu, \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \\ g_\nu &= \frac{\delta g_\nu}{\delta f_1} f_1 + \frac{\delta g_\nu}{\delta f_2} f_2 + \ldots + \frac{\delta g_\nu}{\delta f_\nu} f_\nu, \end{aligned}$$

whence Mr. Sylvester derives the expression, that the substitution operated on the variables

(5) $$\begin{array}{l} k_{1,1}, k_{1,2}, \ldots k_{1,n}, \\ \ldots\ldots\ldots, \\ \ldots\ldots\ldots, \\ k_{n,1}, k_{n,2}, \ldots k_{n,n}, \end{array}$$

induces the substitution operated on the elements

(6) $$\begin{array}{l} \frac{\delta g_1}{\delta f_1}, \frac{\delta g_1}{\delta f_2}, \ldots \frac{\delta g_1}{\delta f_\nu}, \\ \ldots\ldots\ldots, \\ \ldots\ldots\ldots, \\ \frac{\delta g_\nu}{\delta f_1}, \frac{\delta g_\nu}{\delta f_2}, \ldots \frac{\delta g_\nu}{\delta f_\nu}, \end{array}$$

Let the determinant of the applied substitution (which must not vanish), be denoted by K, the corresponding minors $\frac{\delta K}{\delta K_{a,b}}$ by $K_{a,b}$; then the substitution

(7) $$\begin{array}{l} \frac{K_{1,1}}{K}, \frac{K_{1,2}}{K}, \ldots \frac{K_{1,n}}{K}, \\ \ldots\ldots\ldots, \\ \ldots\ldots\ldots, \\ \frac{K_{n,1}}{K}, \frac{K_{n,2}}{K}, \ldots \frac{K_{n,n}}{K}, \end{array}$$

is said to be *contrary* to the original substitution (5). These things established, Mr. Sylvester gives a general theorem couched in the following terms:

In a prepared form two contrary substitutions operated on the variables induce two contrary substitutions operated on the elements.

Mr. Sylvester having proved this remarkable truth by ascending from binary forms to ternary, from these to quaternary and so on, the present paper will contain a demonstration that embraces the whole theorem in one grasp.

In order to make use of the substitution (7), let the n variables $x_1, x_2, \ldots x_n$ be made equal to the following linear functions of n new variables $z_1, z_2, \ldots z_n$,

(8) $$x_a = \frac{K_{a,1}}{K} z_1 + \frac{K_{a,2}}{K} z_2 + \ldots + \frac{K_{a,n}}{K} z_n,$$

according to which the form $F(x_1, x_2, \ldots x_n)$ will be changed into the likewise prepared form

(9) $$H(z_1, z_2, \ldots z_n) = \sqrt{\pi_1} h_1 Z_1 + \sqrt{\pi_2} h_2 Z_2 + \ldots + \sqrt{\pi_\nu} h_\nu Z_\nu,$$

that contains the constants $h_1, h_2, \ldots h_\nu$. Now we may observe that, on solving the system (8) with respect to the quantities z_b, there arise the equations

(10) $$z_b = k_{1,b}x_1 + k_{2,b}x_2 + \ldots + k_{n,b}x_n\,,$$

where the system of coefficients is got from (5) by interchanging the horizontal and vertical lines with each other, or by transposition. Consequently, the form $H(z_1, z_2, \ldots z_n)$ is transformed into the form $F(x_2, x_2, \ldots, x_n)$ by the substitution

(11) $$\begin{array}{l} k_{1,1},\, k_{2,1},\, \ldots\ldots k_{n,1}, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;, \\ k_{1,n},\, k_{2,n},\, \ldots\ldots k_{n,n}, \end{array}$$

which is the substitution (5) transposed; wherefore, if we form the equations

(12) $$\begin{array}{l} f_1 = \frac{\delta f_1}{\delta h_1} h_1 + \frac{\delta f_1}{\delta h_2} h_2 + \ldots\ldots + \frac{\delta f_1}{\delta h_\nu} h_\nu, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;, \\ f_\nu = \frac{\delta f_\nu}{\delta h_1} h_1 + \frac{\delta f_\nu}{\delta h_2} h_2 + \ldots\ldots + \frac{\delta f_\nu}{\delta h_\nu} h_\nu, \end{array}$$

it is evident that the coefficient $\frac{\delta f_a}{\delta h_\beta}$ may be formed from the coefficient $\frac{\delta g_a}{\delta f_\beta}$ by changing the substitution (5) into (11), or by changing $k_{a,b}$ into $k_{b,a}$. Considering the effect of the equations (8) upon the form $F(x_1, x_2, \ldots x_n)$ we must say that the substitution (7) operated on the variables induces the substitution operated on the elements

(13) $$\begin{array}{l} \frac{\delta h_1}{\delta f_1}, \frac{\delta h_1}{\delta f_2}, \ldots\ldots \frac{\delta h_1}{\delta f_\nu}, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;, \\ \frac{\delta h_\nu}{\delta f_1}, \frac{\delta h_\nu}{\delta f_2}, \ldots\ldots \frac{\delta h_\nu}{\delta f_\nu}, \end{array}$$

Therefore, it is to be shown, that the substitution (13) is contrary to the substitution (6). But a general property of partial differential coefficients of n functions taken according to n independent variables teaches us that the substitution contrary to (13) is represented with the aid of the partial differential coefficients of the n variables taken according to the n respective functions, as follows:

(14)
$$\begin{array}{l} \frac{\delta f_1}{\delta h_1}, \frac{\delta f_2}{\delta h_1}, \ldots . \frac{\delta f_\nu}{\delta h_1}, \\ \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; , \\ \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; , \\ \frac{\delta f_1}{\delta h_\nu}, \frac{\delta f_2}{\delta h_\nu}, \ldots . \frac{\delta f_\nu}{\delta h_\nu}. \end{array}$$

Of course our theorem only requires us to establish that the substitutions (6) and (14) accord with each other, or that for any combination of the numbers α, β the equation

(15)
$$\frac{\delta f_\alpha}{\delta h_\beta} = \frac{\delta g_\beta}{\delta f_\alpha}$$

is valid. Meanwhile, we have seen that the partial differential coefficient $\frac{\delta g_\alpha}{\delta f_\beta}$ is changed into $\frac{\delta f_\alpha}{\delta h_\beta}$ by changing $k_{a,b}$ into $k_{b,a}$. Consequently, it will suffice to prove, *that the partial differential coefficient* $\frac{\delta g_\alpha}{\delta f_\beta}$ *turns into the partial differential coefficient* $\frac{\delta g_\beta}{\delta f_\alpha}$, *if* $k_{a,b}$ *is changed into* $k_{b,a}$.

Supposing that the forms in question are expressed in the usual manner,

(16) $F(x_1, x_2, \ldots . x_n) = \pi_1 F_1 X_1 + \pi_2 F_2 X_2 + \ldots . \pi_\nu F_\nu X_\nu$,

(17) $G(y_1, y_2, \ldots . y_n) = \pi_1 G_1 Y_1 + \pi_2 G_2 Y_2 + \ldots . \pi_\nu G_\nu Y_\nu$,

(18) $H(z_1, z_2, \ldots . z_n) = \pi_1 H_1 Z_1 + \pi_2 H_2 Z_2 + \ldots . \pi_\nu H_\nu Z_\nu$,

the constants f_α, g_β, h_γ are connected with the constants F_α, G_β, H_γ, by the purely numerical relations,

(19)
$$f_\alpha = \sqrt{\pi_\alpha} F_\alpha, \; g_\beta = \sqrt{\pi_\beta} G_\beta, \; h_\gamma = \sqrt{\pi_\gamma} H_\gamma,$$

so that, instead of (15), we have the equation

(20)
$$\pi_\alpha \frac{\delta F_\alpha}{\delta H_\beta} = \pi_\beta \frac{\delta G_\beta}{\delta F_\alpha}.$$

In order to prove the former, we are going to prove the latter. Taking notice of the fact that the partial differential coefficient $\frac{\delta G_\alpha}{\delta F_\beta}$ turns into $\frac{\delta F_\alpha}{\delta H_\beta}$ by changing $k_{a,b}$ into $k_{b,a}$, the meaning of (20) may be expressed in the words, *that the product* $\pi_\alpha \frac{\delta G_\alpha}{\delta F_\beta}$ *is changed into the product* $\pi_\beta \frac{\delta G_\beta}{\delta F_\alpha}$ *by changing* $k_{a,b}$ *into* $k_{b,a}$.

As it is permitted to regard the quantities Y_β as linear functions of the quantities X_α and reciprocally, by differentiating the equation

$$F(x_1, x_2, \ldots x_n) = G(y_1, y_2, \ldots y_n)$$

first according to F_α, afterwards according to Y_β, we shall find

(21) $$\pi_\alpha X_\alpha = \sum\nolimits_\beta \pi_\beta \frac{\delta G_\beta}{\delta F_\alpha} Y_\beta ; \quad \pi_\alpha \frac{\delta X_\alpha}{\delta Y_\beta} = \pi_\beta \frac{\delta G_\beta}{\delta F_\alpha} .$$

In like manner from the equation $F(x_1, x_2, \ldots x_n) = H(z_1, z_2, \ldots z_n)$ result the equations

(22) $$\sum\nolimits_\alpha \pi_\alpha \frac{\delta F_\alpha}{\delta H_\beta} X_\alpha = \pi_\beta Z_\beta ; \quad \pi_\alpha \frac{\delta F_\alpha}{\delta H_\beta} = \pi_\beta \frac{\delta Z_\beta}{\delta X_\alpha} .$$

Whence it is evident, that the equation (20) will be proved true if the equation

(23) $$\pi_\alpha \frac{\delta X_\alpha}{\delta Y_\beta} = \pi_\beta \frac{\delta Z_\beta}{\delta X_\alpha}$$

is proved to hold good.

This equation containing no trace of the respective forms, let us denote by $t_1, t_2, \ldots t_n$ a set of n new independent variables, with which we form the expression

(24) $$t_1 x_1 + t_2 x_2 + \ldots + t_n x_n ,$$

which is to be elevated to the pth power. By the aid of the equation (2) and of the definition

(25) $$u_b = k_{1,b} t_1 + k_{2,b} t_2 + \ldots + k_{n,b} t_n$$

the expression (24) assumes the following shape

(26) $$u_1 y_1 + u_2 y_2 + \ldots + u_n y_n .$$

Hence arise by means of the previously introduced notation the expressions

(27) $$(t_1 x_1 + t_2 x_2 + \ldots t_n x_n)^p = \pi_1 T_1 X_1 + \pi_2 T_2 X_2 + \ldots + \pi_\nu T_\nu X_\nu ,$$

(28) $$(u_1 y_1 + u_2 y_2 + \ldots + u_n y_n)^p = \pi_1 U_1 Y_1 + \pi_2 U_2 Y_2 + \ldots + \pi_\nu U_\nu Y_\nu .$$

Considering that X_α and Y_β as well as T_α and U_β depend linearly on one another, we may differentiate the equivalent expressions (27) and (28), first according to T_α, afterwards to Y_β, and get

(29) $$\pi_\alpha X_\alpha = \sum\nolimits_\beta \pi_\beta \frac{\delta U_\beta}{\delta T_\alpha} Y_\beta ; \quad \pi_\alpha \frac{\delta X_\alpha}{\delta Y_\beta} = \pi_\beta \frac{\delta U_\beta}{\delta T_\alpha} .$$

But as by virtue of (25) and (10) the variables u_b depend upon the variables t_a in the same way as the variables z_b depend upon the variables x_a, we conclude that the quantities U_β must be the same linear functions of the quantities T_α as the quantities Z_β are of the quantities X_α, and that consequently the partial differential coefficients $\frac{\delta U_\beta}{\delta T_\alpha}$ and $\frac{\delta Z_\beta}{\delta X_\alpha}$ denote the same thing. Hence it follows that the second equation in (29) produces the equation (23) as was to be proved, and thus the demonstration of the proposed theorem is accomplished.

NOTE ON THE THEOREM CONTAINED IN PROFESSOR LIPSCHITZ'S PAPER.

By J. J. Sylvester.

I think it may be useful to state the principle to which the theorem demonstrated in the preceding paper leads, in the shape in which it has always presented itself to my mind, but which I found difficult to express when writing under the constraint of a foreign language.

It amounts simply to the statement that in a *prepared* form just as the variables $(x, y, z, \ldots)$ are contragredient to their symbolic inverses $\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \ldots\right)$ so the coefficients $(a, b, c, \ldots)$ are contragredient to theirs $\left(\frac{d}{da}, \frac{d}{db}, \frac{d}{dc}, \ldots\right)$; the latter statement in fact includes the former inasmuch as the so-called variables may be regarded as the coefficients of an auxiliary linear form.

In applying this principle it is expedient to enlarge our conception of invariants, covariants, etc., and to predicate invariance of functions not only of quantities ordinarily so termed, but of their symbolic inverses, or of functions in which quantities and operators enter conjointly.* To draw the conclu-

* It was through this idea that I was originally led to an intuitive perception of the theorems concerning the prepared form. For suppose $(a, b, c, \ldots l \between x, y, z)^n$ to be any prepared form; then if x', y', z'; x'', y'', z'' are cogredient with x, y, z, and we operate upon the given form with

$$(\dot{a}, \dot{b}, \dot{c}, \ldots \dot{l} \between y'z'' - y''z', z'x'' - z''x', x'y'' - y''x')^n,$$

the result is the nth power of the determinant $\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}$ which is a covariant. Hence we may conclude that the operator is a covariant; just as, if a covariant multiplied by any form is a covariant, we may conclude that the multiplier must be so too. Consequently $(\dot{a}, \dot{b}, \dot{c}, \ldots \dot{l} \between x, y, z)^n$ is a contravariant, and the same reasoning will apply whatever may be the number of variables. I originally used two forms, one the ordinary form for $(a, b, c, \ldots l \between x, y, z, \ldots)$ and the other the ordinary form *divested* of its numerical coefficients for $(\dot{a}, \ldots \between y'z'' - y''z', \ldots)^n$, and of course with the same result.

The reasoning is perhaps not absolutely rigorous, but sufficiently so to bring conviction of the fact to be established. Of course when we have proved that $(\dot{a}, \dot{b}, \dot{c}, \ldots \dot{l} \between x, y, z)^n$ is a contravariant, it follows more generally that if $(A, B, C, \ldots \between x, y, z, \ldots)^N$ is a covariant $(\dot{A}, \dot{B}, \dot{C}, \ldots \between x, y, z, \ldots)^N$, where $\dot{A}, \dot{B}, \dot{C}, \ldots$ are the same functions of $\dot{a}, \dot{b}, \dot{c}, \ldots$ as $A, B, C, \ldots$ are of $a, b, c, \ldots$, will be a contravariant and *vice versa.*

sions which flow from this conception, we have only to add the rule that all *combinations* of invariants, or of covariants, or of contravariants, etc., are themselves invariants, or covariants, or contravariants, etc., respectively. We may then state that the effect of substituting in a covariant or contravariant (to a prepared form) in place of the variables, or in place of the coefficients, the symbolic inverses of the one or the other, is to reverse their character and convert covariants into contravariants and *vice versa*, leaving of course the character of invariants unaltered; and I may remark incidentally that we are thus provided with a means of making any two *invariants* operate on each other so as to produce a third, a mode of operation which was not possible previous to the introduction of the prepared form.*

Moreover, the word combination must be taken in its widest sense, as there is more than one *mode* of combination possible. For example, if F, G, H are covariantive and Φ a contravariantive function of $(a, b, c, \ldots; x, y, z, \ldots)$, where $a, b, c, \ldots$ are the coefficients and $x, y, z, \ldots$ the variables of a prepared form, we have of course $\Phi\left(\frac{d}{da}, \frac{d}{db}, \ldots; \ x, y, \ldots\right)F$ and $\Phi\left(a, b, \ldots; \frac{d}{dx}, \frac{d}{dy}, \ldots\right)F$ and $G\left(\frac{d}{da}, \frac{d}{db}, \ldots; \frac{d}{dx}, \frac{d}{dy}, \ldots\right)F$ all of them covariants. This may be termed an external mode of combination, but we shall equally have covariants derived by an internal mode of combination, *ex. gr.* $\Phi\left(\frac{dF}{da}, \frac{dF}{db}, \ldots; x, y, \ldots\right)$, $\Phi\left(a, b, \ldots; \frac{dF}{dx}, \frac{dF}{dy}, \ldots\right)$, $H\left(\frac{dF}{da}, \frac{dF}{db}, \ldots; \frac{dG}{dx}, \frac{dG}{dy}, \ldots\right)$ will also be covariants.

So again it may be observed that these modes of combination admit of being applied in more than one way: thus, to confine ourselves for a moment to the case of two forms, their external operation on each other may be simple, or concurrent, or reciprocal: simple when in *one* of them one set of quantities are converted into operators, concurrent when both sets are so converted, but reciprocal when in one of the two forms the variables and in the other the coefficients undergo such conversion. As an example,

*The case may be stated thus: previous to the introduction of the *prepared form*, invariants of systems could be made to operate upon invariants solely through the instrumentality of the coefficients of the linear forms of the system; since its introduction the same operation may be made to take effect through the instrumentality of the coefficients of all the forms, linear or non-linear, indiscriminately. The first named mode of operation is equivalent to the hyperdeterminantive method, which includes that of *Ueberschiebung;* the latter transcends the sphere of hyperdeterminants.

suppose we take the prepared form $ax^3 + \ldots + dy^3$, and its skew covariant $(a^2d + \ldots)\, x^3 + \ldots - (ad^2 + \ldots)\, y^3$. We may combine the contravariant $\left(\frac{d}{da}\, x^3 + \ldots + \frac{d}{dd}\, y^3\right)^2$ with the contravariant $\left(a^2d + \ldots\right)\left(\frac{d}{dx}\right)^3 + \ldots - \left(ad^2 + \ldots\right)\left(\frac{d}{dy}\right)^3$, and the result will be a numerical multiple of the contravariant $dx^3 + \ldots - ay^3$. If, in the above instance, we denote the square of the primitive and its skew covariant according to their degree and order by $2 \cdot 6$, $3 \cdot 3$ respectively, we may explain their mutual action stenographically by saying that $\dot{2} \cdot 6$ and $3 \cdot \dot{3}$ have acted reciprocally on each other, the *dot* signifying that the quantities typified by the number so marked have been replaced by their symbolic inverses; we cannot well represent this mutual action by writing $\dot{2} \cdot 6 * 3 \cdot \dot{3}$ or $3 \cdot \dot{3} * \dot{2} \cdot 6$, but may employ for the purpose $\begin{matrix} \dot{2} \cdot 6 \\ *\,. \\ 3 \cdot \dot{3} \end{matrix}$.
So from the square of a quintic $2 \cdot 10$ and its linear covariant $5 \cdot 1$ we may derive by reciprocal action $\begin{matrix} \dot{2} \cdot 10 \\ *\,. \\ 5 \cdot \dot{1} \end{matrix}$, or the contravariant $3 \cdot 9$: or, again, we may take any even number of covariants and cause them to operate in various manners, the variables on the variables and the coefficients on the coefficients, so as to form a closed circuit, as *ex. gr.* with four, we may make the coefficients of the first operate on those of the second, the variables of the second on those of the third, the coefficients of the third on those of the fourth, and the variables of the fourth on those of the first. Thus we have passed from reciprocal to the more general notion of simultaneous or circulatory action between any even number of covariants. And it is not unlikely that further applications may be made of this fertile conception: when dealing with a principle (an intellectual force) as distinguished from a theorem (a mere law), we never can feel sure that its uses are exhausted, or its plastic power spent.

LETTER FROM MR. MUIR TO PROFESSOR SYLVESTER ON THE WORD CONTINUANT.

In reference to the foot-note on page 127 of your Journal of Mathematics, will you allow me the use of a corner to state that the name "*Continuant*," such as it is, was chosen by me before I had had the pleasure of profiting by your papers in the Philosophical Magazine, and when I believed myself to be the first who had lit upon the peculiar determinant-form in question. On being undeceived by finding the discovery attributed in several German writings (see especially *Günther's Darstellung der Näherungswerthe von Kettenbrüchen in independenter Form*) to continental mathematicians, I carefully examined into the matter, testing the validity of the claims made in these writings, and seeking for earlier traces of the discovery. The result of this work convinced me of your undeniable priority, and when next I had occasion to write on the subject (Phil. Mag., 1877) I called attention, in the most pointed manner, to the conclusion I had come to, and gave the necessary reference to your papers. I have not, however, ceased to use the term "*continuant*," (1) because, as an exceedingly suitable and euphonious abbreviation for "*continued-fraction determinant*," it seems to me to be the very word wanted, (2) because, in this way, it is a short literal translation of the equivalent term "*Kettenbruch-Determinante*," which is the received name in Germany, (3) because, though it may be somewhat scant of meaning to a literalist, I cannot but consider it eminently "suggestive," and (4) because doubtless I have still a foster-father's kindly feeling towards the name he has known another's child by.

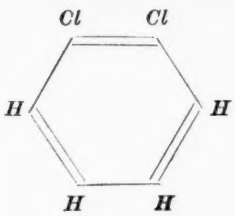
4th September, 1878.

[Reasons 2 and 3 above given appear to afford quite a sufficient justification for the use of the word in question. J. J. S.]

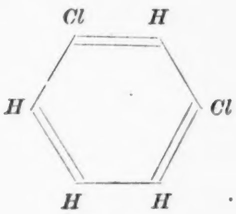
EXTRACT FROM A LETTER OF DR. FRANKLAND TO MR. SYLVESTER.

. . . I WANT now to offer one or two critical remarks upon your papers. . . . At page 65 you notice the anomaly of a *free* bond. My system of notation repudiates all free bonds; but I fear the expression may sometimes have escaped me in describing the transference of an element or compound radical from one compound to another; but I only meant the breaking of the bond for an infinitely short time. I think you also employ the corresponding term "free valence" in this sense at page 69. For free valence, however, I should substitute "latent valence." With very few exceptions, this latent valence is always an even number, hence my hypothesis of two bonds saturating each other (see page 21 of my Lecture Notes).

At page 71, I see you prefer Ladenburg's prism or hexagon to Kekulé's graph for benzol.* It certainly accords better with facts. The facts are 1st. When 1, 5 or 6 atoms of hydrogen in benzol are replaced by *Cl*, *Br*, &c., there are no isomers. 2d. When 2, 3 or 4 atoms are so replaced, there are always 3 isomers and three only. Now, Kekulé's graph exhibits the possibility of four different positions for 2, 3 or 4 atoms of *Cl*, and hence of 4 isomers.



No. 1.



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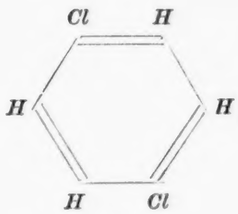

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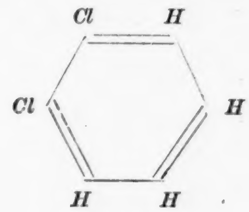

No. 4.

Kekulé does not seem to have noticed the difference between No. 1 and No. 4.

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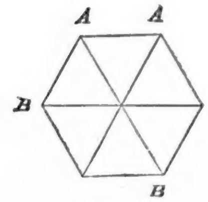
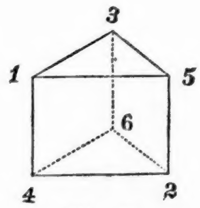
* I did not intend to express any such preference, but merely to state the mathematical fact that the graph of the one is a graph of a covariant belonging to a single biquadratic form which the other is not. The truth is, that chemical graphs correspond to the case of an indefinite repetition of each algebraical form as would necessarily be the case if each chemical atom were capable of playing a different part, or, at all events, of being distinguished in character from every other of the same name with which it may be associated. Mr. Lockyer's recent discoveries seem to favor the notion of this being the case for hydrogen. J. J. S.



No. 5.



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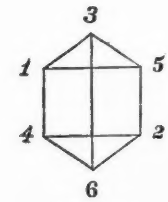

No. 7.

On the other hand, the graph, No. 5, will not answer, because it only allows of two different positions, AA and BB, unless indeed you assign a different value to the cross bonds from that of the lateral ones; but No. 6 gives three different positions, and three only, viz: 1st. Either 1–2, 2–3, 3–4, 4–5, 5–6, or 6–1, for all these pairs are obviously identical. 2d. Either 1–3, 3–5, 5–1, 2–4, 4–6, or 6–2, for these positions again are identical; and 3d. Either 1–4, 2–5, or 3–6, which are also identical. We are now accustomed to call the first set of positions "Ortho," the second "Meta" and the third "Para." I quite agree with you that all graphs may be represented by figures in a plane. Thus, No. 6 may be drawn as No. 7. It is very amusing to see the figure worship of some chemists, and the horror of it in others. Both seem to be unaware that graphic formulæ are only symbols of phenomena, and have no connection whatever with space........

I was at first inclined to smile at your invention of a new set of atoms (atomicules); but, on further consideration, it seems to me that your conception of the constitution of atoms may prove of great value, not in its application to variation of atomicity, but in furnishing an explanation of the behaviour of certain atoms which has long puzzled me and doubtless also other chemists. Carbon affords the best example of what I mean. The strong affinity of carbon for carbon is a quality upon which depends the very existence of nearly every organic compound, and sharply distinguishes these compounds from the almost equally complex siliceous minerals, in which the atoms of silicon are, in no single instance, combined with each other. The most obvious explanation would be that the bonds of carbon are + and — to each other, whilst those of silicon are identical or endowed with energy of the same kind and intensity. But it is a remarkable fact that hitherto, with very few and dubious exceptions, all attempts to establish a difference between the 4 bonds of the carbon atom have failed. Thus, if an atom of carbon be combined with 4 atoms or monad groups a, b, c, d, $\begin{array}{c} b+ \\ | \\ -c-C-\overset{+}{a} \\ | \\ -d \end{array}$ of which a and b

are + and c and d —, it does not matter apparently in what order these bodies are introduced to the carbon atom; for instance, if the body $\overset{+}{a}$ be replaced by a negative body $\overline{e}$, and $\overline{c}$ by a + body $\overset{+}{f}$; and if then in another molecule of $\begin{array}{c} +b \\ | \\ -c-C-\overset{+}{a} \\ | \\ -d \end{array}$, $\overline{c}$ be replaced by $\overline{e}$, and $\overset{+}{a}$ by $\overset{+}{f}$, the two new molecules are identical and not isomeric as might be expected. But if, according to your conception, each bond is tripartite, it could of course exert either + or — energy according to the quality of the atom or group presented to it.

The application of the theory of atomicules to explain latent bonds is of course perfectly legitimate, but I cannot see its advantage over the hypothesis of mutual neutralization. It has never occurred to me that there is any more difficulty in thinking of carbonic oxide as $CC=O$ than in conceiving methyl as $\left(\begin{array}{l} CH_3 \\ CH_3 \end{array} \right.$. If two bonds of two separate but perfectly similar atoms of carbon can satisfy each other, why not two bonds in one and the same atom? If you could apply your theory of atomicules to explain the small number of anomalies which have hitherto defied all other theories, chemistry would be greatly indebted to you. Thus, nitric oxide, if written $N=O$, as required by Avogadro's law, violates the law of variation of atomicity, for a perissad element becomes an artiad. If, on the other hand, it be written, as in my notes, $\begin{array}{c} \subset N=O \\ | \\ \subset N=O \end{array}$ it violates Avogadro's law for equal volumes of hydrogen and nitric oxide do not contain an equal number of molecules.

In reference to your remark on p. 83, every chemico-graph is assumed to be possible, but, in many cases, it is known to be incapable of existing under the conditions prevailing at the surface of the earth.

At page 108, you seem to arrive, by a mathematical process, at $\begin{array}{l} H-C-O-H \\ \quad\ \ \| \\ H-C-H \end{array}$ as the graphic formula of aldehyde; but this is the formula assigned by chemists to vinylic alcohol, which appears, however, to be incapable of existence. The formula universally recognized for aldehyde is $\begin{array}{c} H \\ | \\ H-C-H \\ | \\ O=C-H \end{array}$ which makes its relations to acetic acid $\begin{array}{c} H \\ | \\ H-C-H \\ | \\ O=C-O-H \end{array}$ and

alcohol $\begin{array}{c} H \\ | \\ H - C - H \\ | \\ H - C - O - H \\ | \\ H \end{array}$ simpler than yours.*

The law you mention at page 114, "*m n–valent atoms may be replaced by n m–valent ones*," has a very wide application in chemistry.

I do not think that we have any trustworthy evidence of the difference of H atoms mentioned at page 123.

I cannot, like Mr. Mallet, conceive of "centres of force," or indeed of force at all, apart from matter.

In reference to the origin of the theory of atomicity, I send you, by book post, a copy of a paper of mine presented to the Royal Society, on May 10th 1852, in which, at page 438, *et seq.*, you will find what, so far as I know, was the first enunciation of the theory of atomicity. After some preliminary explanations the law is expressed as follows, at page 440: "Without offering any hypothesis regarding the cause of this symmetrical grouping of atoms, it is sufficiently evident, from the examples just given, that such a tendency or law prevails, and that, *no matter what the character of the uniting atoms may be, the combining power of the attracting element, if I may be allowed the term, is always satisfied by the same number of these atoms.*" I then go on to apply the theory to the explanation of the constitution of organo-metallic compounds. It was in fact the discovery of these bodies which first forced the law upon my attention; for I was surprised to find that the more methyl or ethyl I combined with a metal, the less room was there for O, Cl, &c., although the energy with which the ethylised metal combined with these elements was enormously greater than before ethylisation, some of the ethylised metals being even spontaneously inflammable. If you remember that what I call "the uniting atoms" were monads, and that nearly all chemists then regarded the atom of oxygen (with the atomic weight of 8) as equivalent to one atom of hydrogen, you will readily understand the examples I give. The application of the theory to carbon compounds generally, was made by Kolbe and myself in December, 1856 (see Ann. der Chemie. und Pharm., Bd. CI, s. 257), the first word from Kekulé on the subject of atomicity appearing about a year

* If my memory is not in fault, the form I have given is the only graph possible if each carbon atom refers to the same biquadratic, but there seems no reason to suppose that such is the case in nature. (*Vide* foot-note, page 345.) J. J. S.

later. I have ordered my publisher to send a copy of my "Experimental Researches" to the Library of the Johns Hopkins University. If you will kindly refer to pages 145 and 154, you will see what I have said on the subject last year, whilst at page 188 you will find the paragraphs in my Royal Society memoir, and at page 148 the manifesto of Kolbe and myself which discloses the application of the doctrine of atomicity, or atom fixing power, to the compounds of *C*. Look especially at page 150 for the declaration of Kolbe's conversion to the theory. All this happened, as I have said, long before Kekulé took up the matter; but his subsequent work, and especially the later rectification of atomic weights by Cannizzaro, aided the further development of the theory enormously; but it is singular that no subsequent writer on the subject appears to have known of my paper in the Philosophical Transactions.

I trust that you will go on with the consideration of chemical phenomena from a mathematical point of view, for I am convinced that the future progress of chemistry, as an exact science, depends very much indeed upon the alliance of mathematics. . . . . . .

LONDON, *October 13, 1878.*

APPLICATIONS OF GRASSMANN'S EXTENSIVE ALGEBRA.

By Professor Clifford, *University College, London.*

I propose to communicate in a brief form some applications of Grassmann's theory which it seems unlikely that I shall find time to set forth at proper length, though I have waited long for it. Until recently I was unacquainted with the Ausdehnungslehre, and knew only so much of it as is contained in the author's geometrical papers in *Crelle's Journal* and in *Hankel's Lectures on Complex Numbers.* I may, perhaps, therefore be permitted to express my profound admiration of that extraordinary work, and my conviction that its principles will exercise a vast influence upon the future of mathematical science.

The present communication endeavors to determine the place of Quaternions and of what I have elsewhere* called Biquaternions in the more extended system, thereby *explaining* the laws of those algebras in terms of simpler laws. It contains, next, a generalization of them, applicable to any number of dimensions; and a demonstration that the algebra thus obtained is always a compound of quaternion algebras which do not interfere with one another.

On the Relation of Grassmann's Method to Quaternions and Biquaternions; and on the Generalization of these Systems.

Following a suggestion of Professor Sylvester, I call that kind of multiplication in which the sign of the product is reversed by an interchange of two adjacent factors, *polar* multiplication; because the product ab has opposite properties at its two ends, so that $ab = -ba$. The ordinary or commutative multiplication I shall call *scalar*, being that which holds good of scalar numbers. These words answer to Grassmann's *outer* and *inner* multiplication; which names, however, do not describe the multiplication itself, but rather those geometrical circumstances to which it applies.

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* Proceedings of the London Mathematical Society.

Consider now a system of n units $\iota_1, \iota_2, \ldots \iota_n$, such that the multiplication of any two of them is polar; that is, $\iota_r \iota_s = -\iota_s \iota_r$. For geometrical applications we may take these to represent points lying in a flat space of $n-1$ dimensions. A binary product $\iota_r \iota_s$ is then a unit length measured on the line joining the points ι_r, ι_s; a ternary product $\iota_r \iota_s \iota_t$ is a unit area measured on the plane through the three points, and so on. A linear combination of these units, $\Sigma a_r \iota_r$, $= \alpha$ suppose, represents a point in the given flat space of $n-1$ dimensions, according to the principles of the barycentric calculus, as extended in the Ausdehnungslehre of 1844.

In space of three dimensions we may take the four points $\iota_0, \iota_1, \iota_2, \iota_3$ so that $\iota_1, \iota_2, \iota_3$ are at an infinite distance from ι_0 in three directions at right angles to one another.

Now there are two sides to the notion of a product. When we say $2 \times 3 = 6$, we may regard the product 6 as a number derived from the numbers 2 and 3 by a process in which they play similar parts; or we may regard it as derived from the number 3 by the operation of doubling. In the former view 2 and 3 are both numbers; in the latter view 3 is a number, but 2 is an operation, and the two factors play very distinct parts. *The Ausdehnungslehre is founded on the first view; the theory of quaternions on the second.* When a line is regarded as the product of two points, or a parallelogram as the product of its sides, the two factors are things of the same kind and play similar parts. But in such a quaternion equation as $q\rho = \sigma$, where ρ and σ are vectors, the quaternion q is an operation of turning and stretching which converts ρ into σ; it is a thing totally different in kind from the vector ρ. The only way in which the factors q and ρ can be taken to be of the same kind, is to regard ρ as itself a special case of a quaternion, viz: a rectangular versor. But in that case the expression does not receive its full meaning until we suppose a *subject* on which the operations ρ and q can be performed in succession.

The quaternion symbols i, j, k represent, then, *rectangular versors;* that is to say, they are operations which will turn a figure through a right angle in the three coordinate planes respectively. It follows that if either of them is applied twice over to the same figure, it will turn it through two right angles, or *reverse* it; we must therefore have $i^2 = j^2 = k^2 = -1$.

To compare these with the symbols for the four points $\iota_0, \iota_1, \iota_2, \iota_3$, let us suppose that i turns the line $\iota_0 \iota_2$ into $\iota_0 \iota_3$; that j turns $\iota_0 \iota_3$ into $\iota_0 \iota_1$; and that k

turns $\iota_0\iota_1$ into $\iota_0\iota_2$. The turning of $\iota_0\iota_2$ into $\iota_0\iota_3$ is equivalent to a translation along the line at infinity $\iota_2\iota_3$. We may, therefore, write $i=\iota_2\iota_3$, and so $j=\iota_3\iota_1$, $k=\iota_1\iota_2$. Now i turns $\iota_0\iota_2$ into $\iota_0\iota_3$; that is

$$i\,.\,\iota_0\iota_2=\iota_0\iota_3$$

or

$$\begin{aligned}\iota_0\iota_3&=\iota_2\iota_3\,.\,\iota_0\iota_2\\&=-\iota_2^2\,.\,\iota_0\iota_3\,.\end{aligned}$$

We are therefore obliged to write $\iota_2^2=-1$, and in a similar way we may find $\iota_1^2=\iota_3^2=-1$.

This at once enables us to find the rules of multiplication of the *i*, *j*, *k*. Namely, we have

$$\begin{aligned}jk&=\iota_3\iota_1\,.\,\iota_1\iota_2=\iota_2\iota_3=i\\ki&=\iota_1\iota_2\,.\,\iota_2\iota_3=\iota_3\iota_1=j\\ij&=\iota_2\iota_3\,.\,\iota_3\iota_1=\iota_1\iota_2=k\end{aligned}$$

and finally

$$ijk=\iota_2\iota_3\,.\,\iota_3\iota_1\,.\,\iota_1\iota_2=-1\,.$$

In order, therefore, to bring the quaternion algebra within that of the Ausdehnungslehre, we have to make the square of each of our units equal to -1, as pointed out by Grassmann.* But I venture to differ from his authority in thinking that the quaternion symbols do not in the first place answer to the "Elementargrösse" of the Ausdehnungslehre, but to binary products of them; from which supposition, as we have seen, the laws of their multiplication follow at once.

It is quite true that in process of time the conception of a product as derived from factors of the same kind, and so of the product of two vectors as a thing which might be thought of without regarding them as rectangular versors, grew upon Hamilton's mind, and led to the gradual replacement of the units *i*, *j*, *k* by the more general selective symbols *S* and *V*. To explain the laws of multiplication of *i*, *j*, *k* on this view, we must have recourse to the theory of "Ergänzung," or, which comes to the same thing, *represent* an area *ij* by a vector *k* perpendicular to it. But the explanation in this case is by no means so easy; and it is instructive to observe that the distinction between a quantity and its "Ergänzung," *i. e.* between an area and its representative vector, which, for some purposes it is so convenient to ignore, has to be reïntroduced in physics. Thus Maxwell specially distinguishes the two kinds of vectors which he calls *force* and *flow*, and which in fact are respectively linear functions of the units and of their binary products.

* Math. Annalen.

We have regarded the symbols i, j, k as rectangular versors operating on the quantities $\iota_0\iota_1$, $\iota_0\iota_2$, $\iota_0\iota_3$. These quantities are unit lengths measured anywhere on the axes in the positive directions. They have magnitude, direction, and position, and are thus what I have called *rotors* (short for *rotators*) to distinguish them from *vectors*, which have magnitude and direction but no position. A vector is of the nature of the translation-velocity of a rigid body, or of a couple; it may be represented by a straight line of given length and direction drawn *anywhere*. A rotor is of the nature of the rotation-velocity of a rigid body, or of a force; it belongs to a definite axis. A vector may be represented as the difference of two points of equal weight (the vector ab may be written $b - a$); this is shewn by the principles of the barycentric calculus to represent a point of no weight at infinity. Accordingly the symbols $\iota_1, \iota_2, \iota_3$ may be taken to mean unit vectors along the axes. In fact, if we write $\iota_0 + \iota_r = \alpha_r$, the points α will be situated on the axes at unit distance from the origin, and thus $\iota_r = \alpha_r - \iota_0$ will represent the unit vector from the origin to α_r.

The versors i, j, k will operate on these vectors in the same way as on the rotors $\iota_0\iota_1$, $\iota_0\iota_2$, $\iota_0\iota_3$. We find that $i\iota_2 = \iota_2\iota_3 . \iota_2 = \iota_3$, $j\iota_3 = \iota_1$, $k\iota_1 = \iota_2$. These rules of multiplication coincide with those for i, j, k if we write the latter in place of $\iota_1, \iota_2, \iota_3$. Thus we may use the same symbols to represent unit vectors along the axes and rectangular versors about them. But it is not in any sense true that the vectors $\iota_1, \iota_2, \iota_3$ are identical with the areas $\iota_2\iota_3$, $\iota_3\iota_1$, $\iota_1\iota_2$; it is only sometimes convenient to forget the difference between ι_1 and $\iota_2\iota_3$.

In the elliptic or hyperbolic geometry* of three dimensions, the four points $\iota_0, \iota_1, \iota_2, \iota_3$ must be taken as the vertices of a tetrahedron self-conjugate in regard to the absolute, so that the distance between every two of them is a *quadrant*. The product of four points $\alpha\beta\gamma\delta$ will then consist of three kinds of terms; (1) terms of the fourth order, being $\iota_0\iota_1\iota_2\iota_3$ multiplied by the determinant of the coordinates of the four points, which is proportional to $\sin(\alpha, \beta)$ $\sin(\gamma, \delta)$ $\cos(\alpha\beta, \gamma\delta)$; (2) terms of the second order, resulting from products of the form $\iota_0^2\iota_1\iota_2$, $= -\iota_1\iota_2$; (3) terms of order zero, resulting from products of the form ι_0^4, $\iota_0^2\iota_1^2$. Altogether we may arrange $\alpha\beta\gamma\delta$ in eight terms as follows:

$$\alpha\beta\gamma\delta = a + \Sigma b_{rs}\iota_r\iota_s + c\,\iota_0\iota_1\iota_2\iota_3. \quad [r, s \text{ different.}]$$

And it is now easy to see that the product of any *even* number of linear factors will be of the same form. This form is what I have called a *biquaternion*,

* Dr Klein's names for the geometry of a space of uniform positive or negative curvature. See Proc. Lond. Math. Soc.

and may be easily exhibited as such. Namely, let us write ω for $\iota_0\iota_1\iota_2\iota_3$; then we have

$$i = \iota_2\iota_3 \qquad j = \iota_3\iota_1 \qquad k = \iota_1\iota_2$$
$$\omega i = i\omega = \iota_1\iota_0,\ \omega j = j\omega = \iota_2\iota_0,\ \omega k = k\omega = \iota_3\iota_0$$
$$\omega^2 = 1\,.$$

Therefore, the product of any even number of factors greater than two is a linear function of 1, $i, j, k, \omega, \omega i, \omega j, \omega k$; that is to say, it is of the form $q + \omega r$, where q, r are quaternions. While the multiplication of ω with i, j, k is scalar, its multiplication with $\iota_0, \iota_1, \iota_2, \iota_3$ is polar. The effect of multiplying by ω is to change any system into its polar system in regard to the absolute.

The chief classification of geometric algebras is into those of *odd* and *even* dimensions. The geometry of an elliptic space of n dimensions is the same as the geometry of the points at an infinite distance in a flat or parabolic space of $n+1$ dimensions; the theory of *points* and *rotors* in the former is the same as that of vectors and their products in the latter. Each requires a geometric algebra of $n+1$ units. Thus the algebra of four units, leading as above to biquaternions, is either that of points and rotors in an elliptic space of three dimensions, or of vectors and their products in a flat space of four dimensions. All geometric algebras having an even number of units are closely analogous to it; of these I would point out particularly that of two units, belonging to the elliptic geometry of one dimension or to the theory of vectors in a plane. Let the units be ι_2, ι_3; then a product of any even number of linear functions must be of the form $a + b\iota_2\iota_3$. Let $i = \iota_2\iota_3$, then $i^2 = -1$; and such an even product is the ordinary complex number $a + bi$. In the method of Gauss every vector in the plane is represented by means of its ratio to the unit vector ι_2, that is to say, ι_2 and ι_3 are replaced by 1 and i. This gives an artificial but highly useful value for the product of two vectors. We might apply a similar interpretation to the algebra of four units, denoting the points $\iota_0, \iota_1, \iota_2, \iota_3$ by the symbols ω, i, j, k, and consequently their polar planes $\omega\iota_0, \omega\iota_1, \omega\iota_2, \omega\iota_3$ by the symbols $1, \omega i, \omega j, \omega k$; but I am not aware that any useful results would follow from this imitation of Gauss's plane of numbers.

Rules of Multiplication in an Algebra of n units.

In general, if we consider an algebra of n units, $\iota_1, \iota_2, \ldots \iota_n$, such that $\iota_r^2 = -1$, $\iota_r\iota_s = -\iota_s\iota_r$, a product of m linear factors will contain terms which are all of even order if m is even, and all of odd order if m is odd; for the

substitution of -1 for any square factor of a term reduces the order of the term by 2.

A product of m units, all different, multiplied by any scalar is called a *term* of the order m. The sum of several terms of order m, each multiplied by a scalar, is a *form* of order m. The sum of several forms of different orders is a *quantity* and an even quantity when the forms are all of even order, an odd quantity when they are all of odd order. Thus the multiplication of linear functions of the units leads only to even quantities and odd quantities.

The square of a term of the m^{th} order is $+1$ or -1 according as the integer part of $\frac{1}{2}(m+1)$ is even or odd. For the product $\iota_1\iota_2\ldots\iota_m\iota_1\iota_2\ldots\iota_m$ is transformed into $\iota_1^2\iota_2^2\ldots\iota_m^2$ by $\frac{1}{2}m(m-1)$ changes of consecutive factors, and therefore equals ± 1 according as $\frac{1}{2}m(m+1)$ is even or odd; which is equivalent to the rule stated.

The multiplication of a term P of order m by a term Q of order n, having k factors common, is scalar or polar according as $mn-k^2$ is even or odd. Let $P = CP'$ and $Q = CQ'$, where C, P', Q' have no common factor; then the steps from $CP'CQ'$ to $CP'Q'C$, $CQ'P'C$, $CQ'CP'$ require respectively $k(n-k)$, $(m-k)(n-k)$, $k(m-k)$ changes of consecutive factors; and the sum of these quantities is even or odd as $mn-k^2$ is.

The following cases are worth noticing:

(1) When two terms have no factor common, their multiplication is scalar except when they are both of odd order. (Case $k=0$).

(2) The multiplication of two even terms is scalar or polar according as the number of common factors is even or odd.

(3) If one of two terms is a factor in the other, the multiplication is scalar except when the first is odd and the second even.

Theory of Algebras with an odd number of units.

When the number of units is $n = 2m+1$, there are n terms of the order $n-1$, and all terms of even order can be expressed by means of these. For the product of any two of these terms is of the second order, since they must have $n-2$ factors common. We obtain in this way all the terms of the second order; and from them we can build up the terms of the fourth, sixth

orders, etc. Let the product of all the units $\iota_1 \iota_2 \ldots \iota_n$ be called ω, then these terms of the order $n-1$ shall be defined by the equations $k_r = \omega\iota_r$. It will follow that $k_1 k_2 \ldots k_n = \mp 1$ according as m is even or odd, or, which is the same thing, according as the squares of the k are $+1$ or -1. By means of this formula, terms of order higher than m in the k, may be replaced by terms of order not higher than m. The multiplication of the k is always polar.

The terms of even order, regarded as compound units, constitute an algebra which is *linear* in the sense of Professor Peirce, viz: it is such that the product of any two of these terms is again a term of the system. The number of them is $2^{n-1} = 2^{2m}$; for the whole number of terms, odd and even, is $1 + n + \frac{1}{2}n \,.\, \overline{n-1} + \ldots + n + 1 = (1+1)^n = 2^n$, and the number of even terms is clearly equal to the number of odd terms.

I shall call the algebra whose units are the even terms formed with n elementary units $\iota_1 \iota_2 \ldots \iota_n$, the *n-way geometric algebra.* Thus quaternions are the *three-way* algebra. We may regard the units of quaternions as expressed in either of two ways. First, in terms of the elementary units $\iota_1 \iota_2 \iota_3$; they are then $(1, \iota_2\iota_3, \iota_3\iota_1, \iota_1\iota_2)$. Secondly, we may write k_1, k_2 for the terms $\iota_2\iota_3$, $\iota_3\iota_1$, and the system may then be written $(1, k_1, k_2, k_1 k_2)$. In this second form it is identical with the entire algebra of two elementary units, including both odd and even terms.

The five-way algebra depends upon the five terms k_1, k_2, k_3, k_4, k_5 and their products; the number of terms is sixteen. Now *we may obtain the whole of these sixteen terms by multiplying the quaternion set*

$$(1, k_1, k_2, k_1 k_2)$$

by this other quaternion set

$$(1, k_4 k_5, k_5 k_3, k_3 k_4).$$

For each of the sixteen products so obtained is a term of the even five-way algebra, and the products are all distinct. Moreover, *the two quaternion sets are commutative with one another.* For since the k multiply in the polar manner, we may regard them as elementary units for this purpose; now the terms in the second set are all even, and no term in one set has a factor common with any term in the other set.

In the language of Professor Peirce, then, the five-way algebra is a compound of two quaternion algebras, which do not in any way interfere, because the units of one are commutative in regard to those of the other. A quantity

in the five-way algebra is in fact a quaternion $\omega + ix + jy + kz$, whose coefficients $\omega\ x\ y\ z$ are themselves quaternions of another set of units $(1, i_1, j_1, k_1)$, the i_1, j_1, k_1, being commutative with i, j, k.

I shall now extend this proposition, and shew that *the* $(2m+1)$*-way algebra is a compound of* m *quaternion algebras, the units of which are commutative with one another.* To this end let us write $p_0 = k_1 k_2$, and then

$$\begin{array}{ll} p_1 = k_1 k_2 k_6 k_7 = p_0 k_6 k_7 & q_1 = k_3 k_4 k_5 \\ p_2 = p_1 k_{10} k_{11} & q_2 = q_1 k_8 k_9 \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ p_r = p_{r-1} k_{4r+2} k_{4r+3} & q_r = q_{r-1} k_{4r} k_{4r+1}. \end{array}$$

Consider now the quaternion sets

$$\begin{array}{l} 1, k_1, k_2, k_1 k_2 \\ 1, k_4 k_5, k_5 k_3, k_3 k_4 \\ 1, p_0 k_6, p_0 k_7, k_6 k_7 \\ 1, q_1 k_8, q_1 k_9, k_8 k_9 \\ 1, p_1 k_{10}, p_1 k_{11}, k_{10} k_{11} \\ \cdots\cdots\cdots\cdots\cdots \\ 1, q_{r-1} k_{4r}, q_{r-1} k_{4r+1}, k_{4r} k_{4r+1} \\ 1, p_{r-1} k_{4r+2}, p_{r-1} k_{4r+3}, k_{4r+2} k_{4r+3} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

viz: a p-set and a q-set alternately. I say that if we consider the first m sets of this series, we shall find them to involve $2m+1$ of the k; that the products of m terms, one from each series, constitute 2^{2m} distinct terms, which are therefore identical with the terms of the $(2m+1)$-way algebra; and that the terms in any two sets are commutative with each other. The first two remarks are obvious on inspection; the last also is clear for the case of a p-set and a q-set, because the q-set is of even order in the k, and no factors are common to the two sets. It remains only to examine the case of two p-sets and of two q-sets. Consider the two p-sets

$$\begin{array}{l} 1, p_{r-1} k_{4r+2}, p_{r-1} k_{4r+3}, k_{4r+2} k_{4r+3}, \\ 1, p_{s-1} k_{4s+2}, p_{s-1} k_{4s+3}, k_{4s+2} k_{4s+3}, \end{array}$$

where $s > r$. All the terms of the first set are contained as factors in each of the terms $p_{s-1} k_{4s+2}, p_{s-1} k_{4s+3}$, which are of odd order in the k; consequently, the multiplication is scalar. The term $k_{4s+2} k_{4s+3}$ has no factor common with the first set, and being of even order is commutative in regard to it. Hence the two sets are commutative with one another. Next take the two q-sets

$$\begin{array}{l} 1, q_{r-1} k_{4r}, q_{r-1} k_{4r+1}, k_{4r} k_{4r+1}; \\ 1, q_{s-1} k_{4s}, q_{s-1} k_{4s+1}, k_{4s} k_{4s+1}. \end{array}$$

Here again all the terms of the first set are factors of $q_{s-1}k_{4s}$ and of $q_{s-1}k_{4s+1}$, and they have no factors in common with $k_{4s}k_{4s+1}$; since then all the terms are of even order in the k, the multiplication is scalar. The proposition is therefore proved.

We may set out a formal proof that the 2^{2m} products of m terms, one from each of the first m sets, are all *distinct*, as follows: Suppose this true for the first $m-1$ sets: that is to say, that no two of the products formed from them are either identical or such that their product is $\pm k_1 k_2 \ldots k_{2m-1}$. Let then a, b be two of these products; and let c, d be two terms of the next set. Then we have to prove that ac can neither be equal to $\pm bd$, nor such that the product $acbd$ is $\pm k_1 k_2 \ldots k_{2m-1} k_{2m} k_{2m+1}$. Now if $ac = \pm bd$, multiply both sides by bc; then $ab = \pm cd$. The product cd is one of the terms of the new set; it is either unity, or contains one or both of the new units k_{2m}, k_{2m+1}, so that it cannot be equal to ab. The product $abcd$ cannot be $\pm k_1 \ldots k_{2m+1}$ unless cd is $k_{2m}k_{2m+1}$ and ab is $k_1 k_2 \ldots k_{2m-1}$, which is contrary to the supposition. Hence if the products of the first $m-1$ sets are all distinct for the purposes of the $(2m-1)$-way algebra, the products of the first m sets will be all distinct for the purposes of the $(2m+1)$-way algebra. But it is easy to see that the products of the first two sets are distinct.

Algebras with an even number of units.

Every algebra with $2m$ units is related to the adjacent algebra with $2m-1$ units in precisely the same way as biquaternions are related to quaternions; namely, it is simply that adjacent algebra multiplied by the double algebra $(1, \omega)$ where ω is the product of all the $2m$ units. For clearly all the even terms of the $(2m-1)$-way algebra are also even terms of the $2m$-way algebra, and so also are their products by ω; but these are all distinct from one another, and consequently are *all* the even terms of the $2m$-way algebra.

The multiplication of ω with the k of the $(2m-1)$-way algebra is scalar, because the k are factors in the ω, and they are both even terms.

Hence the $2m$-way algebra is a product of the $(2m-1)$-way algebra with the double algebra $(1, \omega)$, the two sets of units being commutative with one another.

THE MOTION OF A POINT UPON THE SURFACE OF AN ELLIPSOID.

By Thomas Craig, *Fellow of the Johns Hopkins University.*

The ideas contained in the following brief paper were suggested to me in reading the twenty-eighth Chapter of Jacobi's Vorlesungen über Dynamik. A similar investigation may have been given before, but I have never seen anything on the subject.

Let the point be acted upon by a force directed constantly towards the center of the ellipsoid and varying as the distance from the center. Let β, a constant, denote the force at unit's distance from the center, and let α denote the force in the direction of the normal to the surface. We have then for the equations of motion of the point

$$\begin{aligned}
&\frac{d^2x}{dt^2} = \alpha\,\frac{x}{a^2} + \beta x \\
(1)\qquad &\frac{d^2y}{dt^2} = \alpha\,\frac{y}{b^2} + \beta y \\
&\frac{d^2z}{dt^2} = \alpha\,\frac{z}{c^2} + \beta z\,,
\end{aligned}$$

the ellipsoid being given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,.$$

Multiplying equations (1) by $\frac{x}{a^2}$, $\frac{y}{b^2}$, $\frac{z}{c^2}$ respectively, and adding, we obtain

$$\begin{aligned}
\frac{x}{a^2}\frac{d^2x}{dt^2} + \frac{y}{b^2}\frac{d^2y}{dt^2} + \frac{z}{c^2}\frac{d^2z}{dt^2} &= \alpha\left[\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right] \\
&\quad + \beta\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right].
\end{aligned}$$

Calling $\sqrt{p}$ the reciprocal of the length of the perpendicular from the center on the tangent plane at the point x, y, z, we may write this equation

$$(a)\qquad \frac{x}{a^2}\frac{d^2x}{dt^2} + \frac{y}{b^2}\frac{d^2y}{dt^2} + \frac{z}{c^2}\frac{d^2z}{dt^2} = \alpha p + \beta\,.$$

Differentiating the equation of the ellipsoid twice with respect to t, we have

$$\frac{x}{a^2}\frac{dx}{dt} + \frac{y}{b^2}\frac{dy}{dt} + \frac{z}{c^2}\frac{dz}{dt} = 0\,,$$

$$\frac{x}{a^2}\frac{d^2x}{dt^2} + \frac{y}{b^2}\frac{d^2y}{dt^2} + \frac{z}{c^2}\frac{d^2z}{dt^2}$$

$$+ \frac{1}{a^2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{b^2}\left(\frac{dy}{dt}\right)^2 + \frac{1}{c^2}\left(\frac{dz}{dt}\right)^2 = 0;$$

designate by P the last three terms of this last equation, then we have from (a)

(2) $$-P = \alpha p + \beta\,.$$

Again, equations (1) give

$$\left[\frac{1}{a^2}\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{1}{b^2}\frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{1}{c^2}\frac{dz}{dt}\frac{d^2z}{dt^2}\right] = \alpha\left[\frac{x}{a^4}\frac{dx}{dt} + \frac{y}{b^4}\frac{dy}{dt} + \frac{z}{c^4}\frac{dz}{dt}\right]$$

$$+ \beta\left[\frac{x}{a^2}\frac{dx}{dt} + \frac{y}{b^2}\frac{dy}{dt} + \frac{z}{c^2}\frac{dz}{dt}\right];$$

this is simply

(3) $$\frac{dP}{dt} = \alpha\frac{dp}{dt}\,.$$

Eliminating α from (2) and (3), we have

$$\frac{1}{p}\frac{dp}{dt} + \frac{1}{P+\beta}\frac{dP}{dt} = 0\,,$$

the integral of which is

(4) $$p\,(P + \beta) = A = \text{const.}$$

Multiplying equations (1) by $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$ respectively, and adding, gives

$$\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2} = \beta\left[x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right].$$

Integrating this we get

(5) $$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2 = \beta\,(x^2 + y^2 + z^2) + B\,,$$

where $B =$ const. of integration. For $\beta = 0$ or when no force acts towards the center, we have the simple case,

$$s = \sqrt{B}\,.\,t + \text{const.}$$

or the arc varies directly as the time. Any point on the ellipsoid can be given as the intersection of this surface with the two confocals

$$\frac{x^2}{a^2+\lambda_1} + \frac{y^2}{b^2+\lambda_1} + \frac{z^2}{c^2+\lambda_1} = 1$$

$$\frac{x^2}{a^2+\lambda_2} + \frac{y^2}{b^2+\lambda_2} + \frac{z^2}{c^2+\lambda_2} = 1\,.$$

Then we have λ_1 and λ_2 as the elliptic coordinates of the point, $\lambda_1 =$ const. being the equation of one set of lines of curvature and $\lambda_2 =$ const. the equation of the other set, the intersection of any two of which determines the position of a point upon the surface. We have now (Salmon's Geom. of Three Dimen., page 124),

$$x^2 + y^2 + z^2 = a^2 + b^2 + c^2 - (\lambda_1 + \lambda_2);$$

also,

$$(6)\quad ds^2 = dx^2 + dy^2 + dz^2 = \frac{\lambda_1 - \lambda_2}{4}\left[\frac{\lambda_1 d\lambda_1^2}{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)} - \frac{\lambda_2 d\lambda_2^2}{(a^2+\lambda_2)(b^2+\lambda_2)(c^2+\lambda_2)}\right]$$

and for the differentials dx, dy, dz the known values

$$(7)\quad \begin{aligned} dx &= -\frac{1}{2}\frac{a}{\sqrt{(a^2-b^2)(a^2-c^2)}}\left[d\lambda_1\sqrt{\frac{a^2+\lambda_2}{a^2+\lambda_1}} + d\lambda_2\sqrt{\frac{a^2+\lambda_1}{a^2+\lambda_2}}\right] \\ dy &= -\frac{1}{2}\frac{b}{\sqrt{(b^2-c^2)(b^2-a^2)}}\left[d\lambda_1\sqrt{\frac{b^2+\lambda_2}{b^2+\lambda_1}} + d\lambda_2\sqrt{\frac{b^2+\lambda_1}{a^2+\lambda_2}}\right] \\ dz &= -\frac{1}{2}\frac{c}{\sqrt{(c^2-a^2)(c^2-b^2)}}\left[d\lambda_1\sqrt{\frac{c^2+\lambda_2}{c^2+\lambda_1}} + d\lambda_2\sqrt{\frac{c^2+\lambda_1}{c^2+\lambda_2}}\right], \end{aligned}$$

and finally for the perpendicular from the center to the tangent plane

$$(8)\quad \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = \frac{\lambda_1\lambda_2}{abc} = p\,.$$

From the expressions for dx, dy, and dz we can readily obtain

$$(9)\quad \frac{dx^2}{a^2} + \frac{dy^2}{b^2} + \frac{dz^2}{c^2} = \frac{\lambda_1 - \lambda_2}{4}\left[\frac{d\lambda_1^2}{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)} - \frac{d\lambda_2^2}{(a^2+\lambda_2)(b^2+\lambda_2)(c^2+\lambda_2)}\right],$$

the first member of which equation, divided by dt^2, is the quantity P, therefore

$$(10)\quad P = \frac{\lambda_1 - \lambda_2}{4}\left[\frac{\left(\frac{d\lambda_1}{dt}\right)^2}{\Phi} - \frac{\left(\frac{d\lambda_2}{dt}\right)^2}{\Psi}\right];$$

where for convenience we write

$$\Phi = (a^2 + \lambda_1)(b^2 + \lambda_1)(c^2 + \lambda_1)$$
$$\Psi = (a^2 + \lambda_2)(b^2 + \lambda_2)(c^2 + \lambda_2)\,.$$

Substituting for P its value as obtained from equation (4), this last becomes

$$(11)\quad \frac{\lambda_1 - \lambda_2}{4}\left[\frac{1}{\Phi}\left(\frac{d\lambda_1}{dt}\right)^2 - \frac{1}{\Psi}\left(\frac{d\lambda_2}{dt}\right)^2\right] = \frac{D}{\lambda_1\lambda_2} - \beta$$

where $D = abcA$. Equation (5) becomes in elliptic coordinates

$$(12)\quad \frac{\lambda_1 - \lambda_2}{4}\left[\frac{\lambda_1}{\Phi}\left(\frac{d\lambda_1}{dt}\right)^2 - \frac{\lambda_2}{\Psi}\left(\frac{d\lambda_2}{dt}\right)^2\right] = C - \beta\,(\lambda_1 + \lambda_2)\,,$$

where for brevity we write $C = \beta\,(a^2 + b^2 + c^2) + B$. Eliminate t from

equations (11) and (12), and the result will be the differential equation, in elliptic coordinates, of the path of the point upon the ellipsoid: this is

$$\left(\frac{D}{\lambda_1\lambda_2}-\beta\right)\left(\frac{\lambda_1 d\lambda_1^2}{\Phi}-\frac{\lambda_2 d\lambda_2^2}{\Psi}\right)=[C-\beta(\lambda_1+\lambda_2)]\left[\frac{d\lambda_1^2}{\Phi}-\frac{d\lambda_2^2}{\Psi}\right],$$

or

$$(13)\qquad \frac{\sqrt{\lambda_1}d\lambda_1}{\sqrt{\Phi}\,(D-C\lambda_1+\beta\lambda_1^2)}\pm\frac{\sqrt{\lambda_2}d\lambda_2}{\sqrt{\Psi}\,(D-C\lambda_2+\beta\lambda_2^2)}=0,$$

in which the variables are separated. If again we make $\beta=0$, that is, if after the first impulse, no force acts upon the point but that in the direction of the normal, this equation becomes, on making $-\dfrac{D}{C}=\theta^2$,

$$\frac{\sqrt{\lambda_1}d\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)(\theta^2+\lambda_1)}}\pm\frac{\sqrt{\lambda_2}d\lambda_2}{\sqrt{(a^2+\lambda_2)(b^2+\lambda_2)(c^2+\lambda_2)(\theta^2+\lambda_2)}}=0;$$

this is the differential equation of a geodesic upon the ellipsoid.

The integrals of these expressions will be elliptic for $\theta=0$. If we change the variables λ_1 and λ_2 into new ones defined by the equations

$$\lambda_1=\frac{1}{\xi}$$

$$\lambda_2=\frac{1}{\eta}$$

our differential equation becomes

$$(14)\qquad \frac{d\xi}{\sqrt{(a^2\xi+1)(b^2\xi+1)(c^2\xi+1)(D\xi^2-C\xi+\beta)}}$$

$$\pm\frac{d\eta}{\sqrt{(a^2\eta+1)(b^2\eta+1)(c^2\eta+1)(D\eta^2-C\eta+\beta)}}=0.$$

Write $-\dfrac{C}{D}=k$ and $\dfrac{\beta}{D}=m$, and call $-\xi_1,-\xi_2,-\eta_1-\eta_2$ the roots of the quadratic equations

$$\xi^2+k\xi+m=0,$$

$$\eta^2+k\eta+m=0;$$

and further $\dfrac{1}{a^2}=a',\ \dfrac{1}{b^2}=b',\ \dfrac{1}{c^2}=c'$, and our equation can be written

$$(15)\qquad \int\frac{d\xi}{\sqrt{(\xi+a')(\xi+b')(\xi+c')(\xi+\xi_1)(\xi+\xi_2)}}$$

$$\pm\int\frac{d\eta}{\sqrt{(\eta+a')(\eta+b')(\eta+c')(\eta+\eta_1)(\eta+\eta_2)}}=\text{const.}$$

These are not in general reducible to elliptic integrals. If now we find λ_2 as a function of λ_1, say $x_2=f(\lambda_1)\equiv f$ and $\Psi=F(\lambda_1)\equiv F$, then substituting in

either of the equations (11) and (12) we will be able to find the time t as a function of λ_1.

Making this substitution in equation (12), we readily find

$$t = \int \sqrt{\frac{\left[\lambda_1 F - \Phi f \left(\frac{df}{d\lambda_1}\right)^2\right](\lambda_1 - f)}{4\Phi F\left[c - \beta(\lambda_1 + f)\right]}}\, d\lambda_1 .$$

When we can determine the values of f and F from equation (15), we will be able from this last to express the elliptic coordinates λ_1 and λ_2 as functions of the time, a problem which, for the case of a simple pendulum or point constrained to move upon the surface of a sphere, has been solved by M. Hermite in Crelle's Journal, vol. 85.

We have seen that in general the path traced out by the moving point is not a geodesic, but was such a line where there was no central attracting force, or where $\beta = 0$. It is interesting, however, to observe that, for another value of β, the point will move along a geodesic. Suppose that we have two quantities, τ and σ, the former either constant or a function of the time, the latter constant or a function of the arc s. And now make

$$\beta x = \left[\tau + \sigma \frac{ds}{dt}\right] \frac{dx}{dt}, = \beta_1 \frac{dx}{dt}$$

$$\beta y. = \left[\tau + \sigma \frac{ds}{dt}\right] \frac{dy}{dt}, = \beta_1 \frac{dy}{dt}$$

$$\beta z = \left[\tau + \sigma \frac{ds}{dt}\right] \frac{dz}{dt}, = \beta_1 \frac{dz}{dt}.$$

Our equations of motion now assume the forms

$$\frac{d^2x}{dt^2} = \frac{\alpha x}{a^2} + \beta_1 \frac{dx}{dt}$$

$$\frac{d^2y}{dt^2} = \frac{\alpha y}{b^2} + \beta_1 \frac{dy}{dt}$$

$$\frac{d^2z}{dt^2} = \frac{\alpha z}{c^2} + \beta_1 \frac{dz}{dt}.$$

We have as before

$$\left[\frac{1}{a^2}\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{1}{b^2}\frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{1}{c^2}\frac{dz}{dt}\frac{d^2z}{dt^2}\right] = \alpha\left[\frac{x}{a^4}\frac{dx}{dt} + \frac{y}{b^4}\frac{dy}{dt} + \frac{z}{c^4}\frac{dz}{dt}\right]$$
$$+\beta_1\left[\frac{1}{a^2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{b^2}\left(\frac{dy}{dt}\right)^2 + \frac{1}{c^2}\left(\frac{dz}{dt}\right)^2\right],$$

or simply

$$\frac{dP}{dt} = \alpha \frac{dp}{dt}\, dp + 2\beta_1 P.$$

And again,

$$\frac{x}{a^2}\frac{d^2x}{dt^2}+\frac{y}{b^2}\frac{d^2y}{dt^2}+\frac{z}{c^2}\frac{d^2z}{dt^2}=\alpha\left[\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right]$$
$$+\beta_1\left[\frac{x}{a^2}\frac{dx}{dt}+\frac{y}{b^2}\frac{dy}{dt}+\frac{z}{c^2}\frac{dz}{dt}\right],$$

or simply

$$-P=\alpha p.$$

Eliminating α from these equations

$$\frac{1}{p}\frac{dp}{dt}+\frac{1}{P}\frac{dP}{dt}=2\beta_1,$$

from which we have

$$\frac{dp}{p}+\frac{dP}{P}=2\beta_1 dt=2\left(\tau dt+\sigma\frac{ds}{dt}dt\right).$$

Now writing

$$2\int\tau dt=\mathrm{T}$$
$$2\int\sigma ds=\Sigma$$

we have as the integral of this equation

$$pP=C'e^{\mathrm{T}+\Sigma}.$$

We can also readily obtain from the equations of motion the following

$$\frac{ds}{dt}=C''e^{\frac{1}{2}(\mathrm{T}+\Sigma)}.$$

These equations give

$$\frac{1}{C''^2}\left(\frac{ds}{dt}\right)^2=\frac{pP}{C'},$$

which becomes in elliptic coordinates

$$\frac{C'}{C''^2}\left[\frac{\lambda_1-\lambda_2}{4}\left[\frac{\lambda_1}{\Phi}\left(\frac{d\lambda_1}{dt}\right)^2-\frac{\lambda_2}{\Psi}\left(\frac{d\lambda_2}{dt}\right)^2\right]\right]=\frac{\lambda_1\lambda_2}{abc}\frac{\lambda_1-\lambda_2}{4}\left[\frac{1}{\Phi}\left(\frac{d\lambda_1}{dt}\right)^2-\frac{1}{\Psi}\left(\frac{d\lambda_2}{dt}\right)^2\right].$$

Putting $\dfrac{abcC'}{C''}=-\theta^2$, and multiplying through by dt^2, we have after simple reductions

$$\frac{\sqrt{\lambda_1}d\lambda_1}{\sqrt{\Phi}\,(\lambda_1+\theta^2)}\pm\frac{\sqrt{\lambda_2}d\lambda_2}{\sqrt{\Psi}\,(\lambda_2+\theta^2)}=0,$$

or

$$\frac{\sqrt{\lambda_1}d\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)(\theta^2+\lambda_1)}}\pm\frac{\sqrt{\lambda_2}d\lambda_2}{\sqrt{(a^2+\lambda_2)(b^2+\lambda_2)(c^2+\lambda_2)(\theta^2+\lambda_2)}}=0.$$

This is the differential equation of the path of the moving point which is, as we see, a geodesic upon the ellipsoid.

ON A PROBLEM OF ISOMERISM.

By F. Franklin, *Fellow of the Johns Hopkins University.*

In the partial discussion of a problem of isomerism here given, use is made of the following theorem:

Write all the partitions of n which do not contain more than one 1; let each partition containing 1 count as one, and let each partition not containing 1 count as the number of *different* numbers which occur in it; the sum of the numbers thus obtained is the number of partitions of $n-1$.

For example: Partitions of 6.

6	 1
51	 1
42	 2
33	 1
321	 1
222	 1

Total, 7, and there are 7 partitions of 5.

The proof is as follows:

A. It is plain, in the first place, that the number of partitions of $n-1$ is equal to the number of partitions of n which contain 1; for all the partitions of n which contain 1 become, by dropping a 1, partitions of $n-1$, and all the partitions of $n-1$ become, by affixing a 1, partitions of n containing 1.

B. Secondly, the number of partitions of n which contain more than one 1 is equal to the number formed by counting the partitions not containing 1 in the manner above prescribed. For every partition not containing 1 can be converted into a partition containing more than one 1 by replacing any number comprised in it by the same number of 1's; and it is easily seen that by doing this for each of the *different* numbers in each partition not containing 1, we obtain all the partitions containing more than one 1, and that we get no repetitions. That we obtain all is evident from the fact that *any* partition containing more than one 1 can be converted into a partition

not containing 1 by merging the 1's into a single number. That we get no repetitions is clear when we consider, first, that the *same* partition cannot give two identical results, since in each result we have a different number of 1's; and secondly, that if two identical results arose from two *different* partitions, we could, by merging the 1's in each of these identical results, get back to the two different partitions, which is absurd.

Now, the number of partitions of $n-1$ is equal, as we have seen in A, to the number of partitions of n which contain 1; that is, to the number of partitions of n which contain one 1, *plus* the number of partitions of n which contain more than one 1; and we now see from B that this sum is the number obtained by the rule given at the outset.

The problem of isomerism above alluded to is as follows:

Required to find the number of different compounds that can be formed by n m-valent atoms and $(m-2)\,n+2$ univalent atoms; the word "compound" being understood to mean any arrangement, *whether continuous or not*, in which every atom appears, with exactly the number of bonds to which its valence entitles it; it being understood, moreover, that no two univalent atoms are connected with each other.

The problem reduces at once to the following: In how many ways can $n-1$ bonds be distributed among n atoms, no atom having *more* than m bonds attached to it? (Of course, it is understood that both ends of every bond are attached.) A little consideration would show the truth of this; but it can very easily be formally proved, as follows:

Let x be the number of bonds connecting m-valent atoms with each other; then, since the whole number of bonds, counting those which connect m-valent atoms *twice*, is mn, the number of bonds which connect m-valent atoms with univalent atoms—in other words, the number of univalent atoms—is

$$mn-2x.$$

But the number of univalent atoms is required to be $(m-2)\,n+2$. We have, therefore,

$$mn-2x=(m-2)\,n+2,$$

whence

$$x=n-1.$$

If combinations of univalent atoms were not excluded, the solution would be (as is easily seen from the above) equal to the sum of the numbers of ways in which

$$n-1,\ n,\ n+1,\ \dots\ \frac{mn}{2}\left(\text{or } \frac{mn-1}{2}\right)$$

bonds can be distributed among n atoms, no atom having more than m bonds attached to it.

When $m=2$, the problem can be easily solved. Let us take it as first stated, *i. e.*, univalent atoms not combining with each other. For $m=2$, it reduces to the following: In how many ways can $n-1$ bonds be distributed among n atoms, no atom having more than 2 bonds attached to it?

Let us consider first, what partitions of n (*i. e.*, what groups of the n atoms) are admissible. Since $n-1$ atoms are *necessary* in order that we should have $n-1$ bonds, it is plain that all partitions which contain 1 more than once must be excluded; for every such partition corresponds to a case in which there occur more isolated atoms than one, *i. e.*, less than $n-1$ atoms furnished with bonds. And every partition in which a single 1 occurs, corresponds to one and only one compound of the kind required. For, by making a *closed circuit* of atoms to correspond to each number greater than 1 in the partition (and only by so doing) we obtain $n-1$ bonds as required. We have now only to look, in addition, at the partitions in which 1 does not occur. In order that the compound represented by such a partition should be of the kind required, we must make a closed circuit to correspond to each number with one exception, one bond being left out in the exception. Now, any one of the numbers entering into the partition can be chosen as the exception; so that the partition will represent, in all, as many compounds as there are *different* numbers in the partition. To sum up the above: we must form all the partitions of n which do not contain more than one 1; let each of these partitions which contains 1 count as one, and let each partition not containing 1 count as the number of different numbers which occur in it; the sum of the numbers thus obtained is the required number of compounds. Now, this sum has been proved to be equal to the number of partitions of $n-1$, so that we have: the number of different compounds that can be formed by n bivalent atoms and 2 univalent atoms is equal to *the number of partitions of* $n-1$, if the union of the univalent atoms is not allowed.

If the union of the univalent atoms had been allowed, our result would have been *the number of partitions of n.* For we would then have had to find the number of ways in which n bonds can be distributed among n atoms, and add this to the above. It is obvious that this number is the number of partitions of n not containing 1; for if we had an isolated atom we could not have n bonds, so that we can have no partitions containing 1; and for any partition not containing 1, by making a closed circuit of atoms to correspond to each number in it (and in no other way) we obtain n bonds, as required. Now, from A, the number of partitions of n not containing 1 is the excess of the whole number of partitions of n over the whole number of partitions of $n-1$. Hence, adding the number of partitions of n not containing 1 to the whole number of partitions of $n-1$, we obtain for our result the whole number of partitions of n.

NOTE ON INDETERMINATE EXPONENTIAL FORMS.

By F. Franklin, *Fellow of the Johns Hopkins University.*

If y and z are two functions of x, which become each equal to 0 for a particular value of x, y^z has, for that value, an indeterminate form; and its value is obtained through that of its logarithm. But we shall see that it is generally unnecessary to specially investigate the expression, its value being always 1, provided that the ratio of z, or of some finite power of z, to y, is not infinite.

For we know that the value of x^x when $x=0$ is 1; hence we have $y^z=(y^y)^{\frac{z}{y}}=1$, provided $\frac{z}{y}$ is not infinite. If $\frac{z}{y}$ is infinite, suppose $\frac{z^k}{y}=a$, where a is finite; then we have

$$y^z=\left(\frac{z^k}{a}\right)^z=\frac{z^{kz}}{a^z}=\frac{(z^z)^k}{a^z}=1,$$

unless k is infinite. That is, the value of the expression is always unity, provided that the ratio of z, or of some finite power of z, to y, is not infinite. This condition is, I believe, always fulfilled except in some cases when y or z is itself an exponential or logarithmic function; it is, at any rate, generally easy to see at a glance whether or not such is the case.

It follows from the above that the expression ∞^0 is also always equal to unity, with a similar restriction. For, if y^z assumes this form for a particular value of x, put $y = \frac{1}{u}$ and we have $y^z = \left(\frac{1}{u}\right)^z = \frac{1}{u^z}$; and the value of this expression is, by the preceding, always 1, provided that the ratio of z, or of some finite power of z, to u, is not infinite; or, in other words, provided that the product of z, or of some finite power of z, and y, is not infinite.

The examples given in most of the text-books I have seen, come within the above restriction; the following, for instance, are all the examples of the above forms given by Williamson: ex. 1, p. 102: $(\sin x)^{\tan x}$ when $x = 0$. ex. 2, p. 103: $\left(1 + \frac{a}{x}\right)^x$ when $x = 0$; ex. 3, p. 103: $\left(\frac{1}{x}\right)^{\tan x}$ when $x = 0$; ex; 30, p. 108: $\left(\frac{\log x}{x}\right)^{\frac{1}{x}}$ when $x = \infty$; ex. 44, p. 109: $(\sin x)^{\sin x}$ when $x = 0$. In all these cases we at once recognize that the condition above found is fulfilled, and the value of the function consequently 1.

In the following examples, found in Spitz's "Differential- und Integral-rechnung," the condition is not fulfilled, and the expressions require a special investigation: $x^{\frac{1}{1+2\log x}}$ when $x = 0$; $x^{\frac{1}{\log(e^x - 1)}}$ when $x = 0$; $\left(\frac{1}{1-e^x}\right)^{\frac{1}{x}}$ when $x = \infty$.

It may be remarked, in this connection, that if y^z assumes, for a particular value of x, the form 1^∞, its value can be very simply expressed without taking its logarithm. Put $z = \frac{1}{u}$; the value required is the limiting value of $(1 + \Delta y)^{\frac{1}{\Delta u}} = \lim\, (1 + \Delta y)^{\frac{1}{\Delta y}\frac{\Delta y}{\Delta u}} = \lim\, [(1 + \Delta y)^{\frac{1}{\Delta y}}]^{\frac{\Delta y}{\Delta u}} = e^{\frac{dy}{du}}$.

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A SYNOPTICAL TABLE OF THE IRREDUCIBLE INVARIANTS AND COVARIANTS TO A BINARY QUINTIC, WITH A SCHOLIUM ON A THEOREM IN CONDITIONAL HYPERDETERMINANTS.

By J. J. Sylvester.

It is well known that every binary quintic can be expressed, and in only one way, as the sum of three fifth powers of linear functions of its variables, or which is the same thing, as the sum of the fifth powers of three variables connected by a linear equation, or finally, under the form

$$ax^5 + by^5 + cz^5,$$

subject to the equation

$$x + y + z = 0.$$

If ϕ, ψ be any two covariants of a binary quintic in x, y, the most general expression of the covariant produced by their operation on each other through the variables is

$$\left(\dot{x}\frac{d}{dy} - \dot{y}\frac{d}{dx}\right)^i \phi\psi,$$

where i is any positive integer and and $\dot{x}$, $\dot{y}$ $\left(\text{abbreviations for } \frac{\delta}{\delta x}, \frac{\delta}{\delta y}\right)$ operate on ϕ only whilst $\frac{d}{dx}$, $\frac{d}{dy}$ operate on ψ.

Suppose now that ϕ, ψ are expressed as functions, say Φ, Ψ, of x, y, z, between which there exists the linear relation $lx + my + nz = 0$; it may be shown that the preceding expression becomes identical with

$$\begin{vmatrix} l & m & n \\ \dot{x} & \dot{y} & \dot{z} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \end{vmatrix}^i \Phi\Psi,$$

where x, y, z are to be treated as independent variables. In the present case, therefore, writing

$$(\dot{y} - \dot{z})\frac{d}{dx} + (\dot{z} - \dot{x})\frac{d}{dy} + (\dot{x} - \dot{y})\frac{d}{dz} = \Lambda,$$

$\Lambda^i\Phi\Psi$, or (which will be more convenient for writing) $\Psi\Lambda^i\Phi$ will represent the covariant derived from the alliance of Φ and Ψ.

The twenty-three irreducibles of the quintic may be arranged in the following partially symmetrical order, which is that which I shall adopt as the order of their successive deduction: the first figure denotes the degree in the coefficients, the second the order in the variables.*

	0·4		2·2		4·0					
	1·5		3·3		5·1					
	2·6	3·5	4·4	5·3	6·2	7·1	8·0			
		4·6		6·4		8·2				
3·9		5·7		7·5		9·3		11·1	12·0	
									13·1	
										18·0

There will be two sources of indeterminateness in the expressions obtained for these forms, one universal, arising from the arbitrary addition

* Comparing this arrangement to the distribution of stars in a firmament, it will be observed that there is a tendency to concentration, or the formation of a sort of milky-way, in the zone situated towards the centre, consisting of three bands which comprise between them 15 out of 23, the total number of forms. This phenomenon becomes very much more distinctly marked in the distribution of the 124 irreducible forms appertaining to the septic, the *corrected* table of which I anticipate will have appeared, about simultaneously with the publication of this, in the *Comptes Rendus*. The table previously given in that journal for the seventhic is affected with some inaccuracies chiefly arising from arithmetical errors of calculation, as I made the computation hurriedly and on the point of leaving England for this continent, and also, in part, from the existence of some errors in the table of the reduced generating function, which I accepted, without sufficient examination, as the basis of my work. It may perhaps be worthy of notice that, if we add a unit to the ordinarily received number of irreducible forms in each case (which it is proper to do, since an absolute number is an invariant of the order zero), the numbers of the irreducibles for the 1st, 3d, 5th and 7th orders become 2, 5, 24, 125 respectively. As I am about to compute the irreducibles for the 9th order, we shall soon be in a position to ascertain whether the law indicated in this progression has any foundation in nature: if so, the number for that case should be 626, or thereabouts, but it is not unlikely that the fact of 9 being a composite number may have a tendency to affect the result, probably in the direction of decrease. For binary quantics of the even orders 0, 2, 4, 6, 8 the number of irreducible covariants is 1, 3, 6, 27, 70 respectively (for the last see *Comptes Rendus*, June 24, 1878), which appear to indicate a geometrical progression with the common ratio 3, subject to diminution for higher powers of 2 entering into the order of the quantic.

$(x+y+z)\,M$, the other special to those forms (such as $13\cdot 1$) which can be obtained by the multiplication of lower forms (as $8\cdot 0$, $5\cdot 1$). Our object must be to seek in all cases the simplest expressions that can be obtained.

$2\cdot 2 = 1\cdot 5\,\Lambda^4\,1\cdot 5 \equiv \Sigma\,(abxy + acxz) \equiv \Sigma abxy.$

I use the sign of equivalence to signify that numerical common multipliers are to be rejected.

$4\cdot 0 = 2\cdot 2\,\Lambda^2\,2\cdot 2$

$$= \Sigma\,(\dot{y}-\dot{z})(\dot{z}-\dot{x})(abxy + acxz + bcyz)\frac{d}{dx}\cdot\frac{d}{dy}\,(abxy + acxz + bcyz)$$

$$= \Sigma\,(-ab + ac + bc)\,ab \equiv a^2b^2 + b^2c^2 + c^2a^2 - 2abc\,(a+b+c)$$

$1\cdot 5 = ax^5 + by^5 + cz^5$

$3\cdot 3 = 2\cdot 2\,\Lambda^2\,1\cdot 5 \equiv \Sigma\,ax^3\,(\dot{y}-\dot{z})^2\,(abxy + bcyz + cazx) \equiv abc\Sigma x^3.$

Since $x^2 + y^2 + z^2 = 3xyz + (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$ we have

(*bis*) $\qquad 3\cdot 3 = abcxyz$

$5\cdot 1 = 3\cdot 3\,\Lambda^2\,2\cdot 2 \equiv abc\Sigma\;x(\dot{y}-\dot{z})^2\,(abxy + bcyz + cazx) \equiv abc\Sigma bcx$

$2\cdot 6 = 1\cdot 6\,\Lambda^2\,1\cdot 5 \equiv \Sigma ax^3\,(\dot{y}-\dot{z})^4\,(ax^5 + by^5 + cz^5) \equiv \Sigma ax^3\,(by^3 + cz^3) \equiv \Sigma ab\;x^3y^3$

$3\cdot 5 = 2\cdot 2\,\Lambda\;1\cdot 5 = \Sigma\,(aby + acz)(\dot{y}-\dot{z})(ax^5 + by^5 + cz^5)$

$$\equiv \Sigma\,(aby + acz)(by^4 - cz^4) = \Sigma a(b^2y^5 - c^2z^5) + abc\Sigma\,(zy^4 - yz^4)$$

$4\cdot 4 = 3\cdot 3\,\Lambda^2\,1\cdot 5 \equiv abc\Sigma x\;(\dot{y}-\dot{z})^2\,(ax^5 + by^5 + cz^5) \equiv abc\Sigma\,(bxy^3 + cxz^3)$

$$= abc\Sigma\,[(ax^3 + by^3 + cz^3)\,x - ax^4] \equiv abc\Sigma ax^4$$

$5\cdot 3 = 2\cdot 2\,\Lambda\;3\cdot 3 = \Sigma\,(aby + acz)(\dot{y}-\dot{z})\;abc\,(x^3 + y^3 + z^3)$

$$\equiv abc\Sigma\,(y^2 - z^2)(aby + acz)^* \equiv abc\Sigma ax\,(by^2 - cz^2)$$

$6\cdot 2 = 3\cdot 3\,\Lambda^2\,3\cdot 3 \equiv a^2b^2c^2\Sigma x\;(\dot{y}-\dot{z})^2\,(x^3 + y^3 + z^3) \equiv a^2b^2c^2\Sigma\,(xy + xz)$

$$\equiv a^2b^2c^2\,(xy + yz + zx) \equiv a^2b^2c^2\,(x^2 + y^2 + z^2)$$

$7\cdot 1 = 4\cdot 4\,\Lambda^4\,3\cdot 5 \equiv abc\Sigma\,\{(b+c)\,\dot{x}^4\Sigma\,[(ab^2y^5 - ac^2z^5) + abc\,(zy^4 - yz^4)\}$

$$\equiv a^2b^2c^2\Sigma\,(b+c)\,x^4\Sigma\,(zy^4 - yz^4) = a^2b^2c^2\,\Sigma a\,(y-z)$$

$8\cdot 0 = 4\cdot 4\,\Lambda^4\,4\cdot 4 \equiv a^2b^2c^2\Sigma ax\;(\dot{y}-\dot{z})^3\,(ax^4 + by^4 + cz^4) \equiv a^2b^2c^2\,(ab + ac + bc)$

$4\cdot 6 = 3\cdot 3\,\Lambda\;1\cdot 5 \equiv abc\Sigma x^2\,(\dot{y}-\dot{z})(ax^5 + by^5 + cz^5) \equiv abc\Sigma a\,(y^2 - z^2)\,x^4$

$6\cdot 4 = 2\cdot 2\,\Lambda\;4\cdot 4 = \Sigma(aby + acz)(\dot{y}-\dot{z})\;abc\,(ax^4 + by^4 + cz^4)$

$$= abc\,(aby + acz)(by^3 - cz^3)$$

$$= abc\Sigma(ab^2y^4 + ac^2z^4) + a^2b^2c^2\Sigma\,(zy^3 - yz^3)\,,$$

which, since $\Sigma\,(zy^3 - yz^3)$ contains $x+y+z$,

$$= abc\Sigma\,(c-b)\,a^2x^4$$

$8\cdot 2 = 4\cdot 4\,\Lambda^4\,4\cdot 6 = a^2b^2c^2\Sigma a\;(\dot{y}-\dot{z})^4\,\{\Sigma a\,(y^2 - z^2)\,x^4\} = a^2b^2c^2\Sigma ab\;(x^2 - y^2)$

---

* For $y^2 - z^2$ I substitute $xz - xy$.

$$
\begin{aligned}
3\cdot 9 = 2\cdot 6 \wedge 1\cdot 5 &\equiv \Sigma(abx^2y^3 + acx^2z^3)(\dot{y} - \dot{z})(ax^5 + by^5 + cz^5) \\
&\equiv \Sigma(abx^2y^3 + acx^2z^3)(by^4 - cz^4) \\
&= \Sigma ax^2(b^2y^7 - c^2z^7) + abcx^2y^2z^2\Sigma(zy^2 - yz^2)\ *
\end{aligned}
$$

$$5\cdot 7 = 4\cdot 4 \wedge 1\cdot 5 \equiv abc\Sigma ax^3(\dot{y} - \dot{z})(ax^5 + by^5 + cz^5) \equiv abc\Sigma ab\,(x - y)\,x^3y^3$$

$$7\cdot 5 = 4\cdot 4 \wedge 3\cdot 3 \equiv a^2b^2c^2\Sigma ax^3(\dot{y} - \dot{z})(x^3 + y^3 + z^3) \equiv a^2b^2c^2\Sigma ax^2\,(y^3 - z^3)$$

$$11\cdot 1 = 5\cdot 1 \wedge 6\cdot 2 \equiv a^3b^3c^3\Sigma bc\,(\dot{y} - \dot{z})(x^2 + y^2 + z^2) \equiv a^3b^3c^3\Sigma bc\,(y - z)$$

$$9\cdot 3 = 6\cdot 2 \wedge 3\cdot 3 \equiv a^3b^3c^3\Sigma x\,(\dot{y} - \dot{z})(x^3 + y^3 + z^3) \equiv a^3b^3c^3(x - y)(y - z)(z - x)$$

$$12\cdot 0 = 6\cdot 2 \wedge^2 6\cdot 2 \equiv a^4b^4c^4\Sigma\,(\dot{y} - \dot{z})\,(x^2 + y^2 + z^2) \equiv a^4b^4c^4$$

$$
\begin{aligned}
13\cdot 1 = 7\cdot 1 \wedge 6\cdot 2 &= a^4b^4c^4\Sigma(b - c)\,(\dot{y} - \dot{z})\,(x^2 + y^2 + z^2) \equiv a^4b^4c^4(b - c)\,(y - z) \\
&= a^4b^4c^4\,(\Sigma\,(by + cz) - \Sigma\,(bz + cy)) \\
&= a^4b^4c^4\,(2\,(ax + by + cz) + (bx + cz + ay)) \equiv a^4b^4c^4\Sigma ax
\end{aligned}
$$

$$
\begin{aligned}
18\cdot 0 = 13\cdot 1 \wedge 5.1 &= a^4b^4c^4\Sigma a\,(\dot{y} - \dot{z})\,(bcx + cay + abz) = a^4b^4c^4\Sigma a\,(c - b) \\
&= a^5b^5c^5\,(a - b)\,(b - c)\,(c - a)\,.
\end{aligned}
$$

$18\cdot 0$ may also be obtained by the operation of $11\cdot 1$ on $7\cdot 1$, or instantaneously as the resultant of $1\cdot 5$, $abcxyz$ and $x + y + z$. In the following table the preceding results are collected; for greater brevity instead of the sign of summation I employ the sign $+$ or $-$ to signify respectively the symmetrical or semi-symmetrical completion of the terms to which it is affixed; m is used to signify abc.

1–2	$abxy + \ :\ (a^2b^2 - 2abc^2) +$
3–5	$ax^5 + \ :\ mx^3 +$, or $mxyz\ :\ mbcx +$
6–12	$\begin{cases} abx^3y^3 + \ :\ a^2bx^5 + myz^4 - \ :\ max^4 + \ : \\ mabxy^2 - \ :\ m^2x^2 + \ :\ m^2bx - \ :\ m^2ab + \end{cases}$
13–15	$max^4y^2 - \ :\ ma^2cx^4 - \ :\ m^2abx^2 -$
16–21	$\begin{cases} ab^2x^2y^7 + mx^2y^4z^3 - \ :\ mabx^4y^3 - \ :\ m^2ax^2y^3 - \ : \\ m^3bcy - \ :\ m^3x^2y - \ :\ m^4 \end{cases}$
22	$m^4ax +$
23	$m^5a^2b -$

I propose, at some future time, to apply a similar method to obtain an explicit representation of the irreducible forms appertinent to the binary seventhic, an arduous undertaking, but one that seems likely to lead to the apperception of new forms of complex symmetry. The primitive may, for that case, be represented by $x^7 + y^7 + z^7 + t^7$, connected by the linear equations

* Possibly this expression may be simplifiable by the addition of a suitable multiple of $x + y + z$.

$(l, m, n, p)(x, y, z, t) = 0$, $(\lambda, \mu, \nu, \pi)(x, y, z, t) = 0$, and Λ, the symbol of alliance will be represented by

$$\begin{vmatrix} \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} & \frac{d}{dt} \\ \dot{x} & \dot{y} & \dot{z} & \dot{t} \\ l & m & n & p \\ \lambda & \mu & \nu & \pi \end{vmatrix}.$$

Every in- and co-variant will then be a rational integer function of x, y, z, t and the six minor determinants, which are the parameters of the line represented by the above two linear equations.

It may be worth while to notice the representations of the irreducible derivatives of the quartic when put under the indeterminate form $ax^4 + by^4 + cz^4$, subject to the relation $x + y + z = 0$. We get

$$2\cdot 0 = 1\cdot 4\,\Lambda^4\, 1\cdot 4 = \Sigma a\,(\dot{y} - \dot{z})^4\,(ax^4 + by^4 + cz^4) \equiv ab + bc + ca\ .$$

$$2\cdot 4 = 1\cdot \dot{4}\,\Lambda^2\, 1\cdot 4 = \Sigma ax^2\,(\dot{y} - \dot{z})^2\,(ax^4 + by^4 + cz^4) = abx^2y^2 + acx^2z^2 + bcy^2z^2$$

$$3\cdot 0 = 1\cdot 4\,\Lambda^4\, 2\cdot 4 = \Sigma a\,(\dot{y} - \dot{z})^4\,(abx^2y^2 + acx^2z^2 + bcy^2z^2) \equiv abc$$

$$3\cdot 6 = 1\cdot 4\,\Lambda\, 2\cdot 4 = \Sigma ax^2\,(\dot{y} - \dot{z})(2\cdot 4)$$

$$= \Sigma\,(a^2bx^5y - a^2cx^5z) + abcxyz\Sigma\,(yz^2 - y^2z)\,.$$

As regards the sextic form, the first idea would be to regard it as the resultant, in respect to one of the variables (say z), of the canonical system discovered by me so long ago,

$$\left.\begin{array}{c} ax^6 + by^6 + cz^6 + mxyz\,(x - y)(y - z)(z - x) \\ x + y + z\,, \end{array}\right\}$$

but this will be found to give rise to expressions for the invariants and covariants of extreme complexity. The representations will, I think, be simplified by adopting the new canonical system

$$\left.\begin{array}{c} x^3 + y^3 + z^3 + 3mxyz \\ ayz + bxz + cyx \end{array}\right\}\begin{array}{l}(1)\\(2)\end{array}$$

and considering the sextic as the resultant of (1) and (2). It will then be found that every covariant proper (calling its order, which is always an even number, 2ε) will still be a resultant of (2) and of some new form in x, y, z of order ε.* The fact of the lowering, by one-half, the order of the form in x, y, z, corresponding to a covariant of any given order in x, y, gives a

* For every quantic of an even order in x, y is a ternary quantic in $x^2 + xy$, $y^2 + yx$, $-xy$, which quantities are proportional to x, y, z connected by the equation $xy + xz + yz = 0$.

great (though it may be not an unbalanced) advantage to the new canonical system over the old. On setting out the equation connecting the four completely symmetrical invariants with the square of the skew one of the sextic, and then making this latter equal to zero, we obtain an equation between three absolute invariants of the sextic which may be regarded as the equation to a surface, the analogue of my Bicorn, the *Nomen Triviale* for the bicuspidal unicursal quartic curve. This surface will divide space into two parts, one corresponding to equations of the sixth order with real, the other with conjugate coefficients, or by real linear substitutions transformable into such, the surface itself being the locus of equations of the recurrent form. The facultative part of space, *i. e.* the part corresponding to the case of real coefficients will then separate into two pairs of regions, one pair belonging to the case of 0 and 4, the other to that of 2 and 6 imaginary roots. By this method, however laborious, the solution of the problem of determining the invariantive criteria of the quality of the roots of the sextic (to borrow a term from the chess table) becomes *forced*, and no other mode of attacking the question appears to me to be practicable; nor can it fail to bring into view a surface possessed of remarkable properties.*

SCHOLIUM.

The mode of representing the covariants to a sextic above employed made it imperative, or at least expedient, to discover a method by aid of which the process of alliance, or hyperdetermination, could be performed upon the representative forms themselves, without eliminating one of the variables by means of the equation of condition, and I have obtained the following very general theorem, which, it will presently be seen, contains a solution of the problem in question, and which, as the first example of conditional alliance, or hyperdetermination, it seems to me desirable to put on record.

Let $\phi, \psi, \ldots, \theta$ be i homogeneous functions of the orders $\alpha, \beta, \ldots, \lambda$ in $i+j$ variables, $x, y, \ldots, t$ being i of them and $u, v, \ldots, z$ the j others, and let the variables be connected by the j homogeneous equations

$$L=0, \quad M=0, \ldots, \quad N=0.$$

Call the Jacobian $\dfrac{d(L, M, \ldots, N)}{d(u, v, \ldots, z)}=\Omega$.

*One may see at a glance that this surface cannot be of a higher order than 7, the integer part of 30 : 4. Possibly however, it may not be so high; there will be no difficulty in finding the actual order by means of the known expression for R^2 (Clebsch, Binäre Formen, p. 299,) in terms of the invariants of even degrees.

Let $\Phi, \Psi, \ldots, \Theta$ be the values of $\phi, \psi, \ldots, \theta$ expressed in terms of $x, y, \ldots, t$ alone, and let

$$\begin{vmatrix} \delta_{x_1}, & \delta_{y_1}, & \ldots, & \delta_{t_1} \\ \delta_{x_2}, & \delta_{y_2}, & \ldots, & \delta_{t_2} \\ \cdot & \cdot & \cdot & \cdot \\ \delta_{x_i}, & \delta_{y_i}, & \ldots, & \delta_{t_i} \end{vmatrix}^q (\Omega_1^\alpha \Omega_2^\beta \ldots \Omega_i^\lambda \Phi_1 \Psi_2 \ldots \Theta_i)$$

be called D, it being understood that the meaning of any subscript, say μ, is to cause the letters $x, y, \ldots, t$ to be changed into $x_\mu, y_\mu, \ldots, t_\mu$. Again let the operative determinant of the $(i+j)$th order written below

$$\begin{vmatrix} \delta_{x_1} & \delta_{y_1} & \ldots\ldots & \delta_{z_1} \\ \delta_{x_2} & \delta_{y_2} & \ldots\ldots & \delta_{z_2} \\ \cdot & \cdot & \cdot & \cdot \\ \delta_{x_i} & \delta_{y_i} & \ldots\ldots & \delta_{z_i} \\ (\delta_x L)_\epsilon & (\delta_y L)_\epsilon & \ldots & (\delta_z L)_\epsilon \\ (\delta_x M)_\epsilon & (\delta_y M)_\epsilon & \ldots & (\delta_z M)_\epsilon \\ \cdot & \cdot & \cdot & \cdot \\ (\delta_x N)_\epsilon & (\delta_y N)_\epsilon & \ldots & (\delta_z N)_\epsilon \end{vmatrix}$$

be called J_ϵ (ϵ being any of the suffixes $1, 2, 3, \ldots, i$) then it will be found that to a numerical factor près

$$(\Omega_1^{\alpha-q} \Omega_2^{\beta-q} \ldots \Omega_i^{\lambda-q})(J_1 + J_2 + \ldots + J_i)^q(\phi_1, \psi_2, \ldots, \theta_i) = D.$$

As a corollary, if the functions $L, M, \ldots, N$ are all linear in respect to $u, v, \ldots, z$, and if in respect to $1, u, v, \ldots, z$ the resultants of $\phi, L, M, \ldots, N$; $\psi, L, M, \ldots, N$; ... are $[\Phi], [\Psi], \ldots, [\Theta]$ (which is what we mean by saying that $\phi, \psi, \ldots, \theta$ represent $[\Phi], [\Psi], \ldots, [\Theta]$), it will be easily seen to follow from the above theorem that the qth alliance of these quantics will be itself *represented by*

$$(J_1 + J_2 + \ldots + J_i)^q(\phi_1, \psi_2, \ldots, \theta_i).^*$$

* This expression may be put under the more compact form J^2, J being a matrix in which the first i lines are the same as those common to $J_1, J_2, \ldots, J_i$, and the last j lines are the sums of the corresponding ones in $J_1, J_2, \ldots, J_i$. Although I had submitted it to a mental process of demonstration (or what seemed such) before sending it to the press, I am not without some little misgiving as to the exactitude of the theorem so far as it regards the higher alliances; for those of the first order it is easily verifiable, and, in that case, it should be noticed that each of the i terms in the expression given by it will reproduce separately (but under quite a distinct form) the value of the Jacobian of $\phi, \psi, \ldots, \theta$ $L, \ldots, N$. Some corresponding simplification in practice, it is not improbable, will apply in the general case, supposing my doubts as to the validity of the theorem to prove unfounded. It is important, and greatly enlarges the horizon of the subject, to remark that, inasmuch as any ternary quadric is linearly transformable into the form $xy + yz + zx$, it will follow that any binary quantic of an even order, with its train of covariants, may be represented by corresponding ternary forms of half their respective orders, combined with a perfectly general final conic, so that *ex. gr.* instead of the form $xy + yz + zx$, useful though it be as an intermediate step in the evolution of the theory, we may substitute the handier and more advantageous one $x^2 + y^2 + z^2$ as the auxiliary quadric.

Thus in the particular case where $x, y, \ldots, t$ becomes x, y and $u, \ldots, z$ becomes z and $L, M, \ldots, N$ becomes the single function $xy + yz + zx$, we see that the qth alliance of the quantics represented by ϕ, ψ will be itself represented by

$$\left\{ \begin{vmatrix} \delta_{x_1} & \delta_{y_1} & \delta_{z_1} \\ \delta_{x_2} & \delta_{y_2} & \delta_{z_2} \\ y_1 + z_1 & z_1 + x_1 & x_1 + y_1 \end{vmatrix} + \begin{vmatrix} \delta_{x_1} & \delta_{y_1} & \delta_{z_1} \\ \delta_{x_2} & \delta_{y_2} & \delta_{z_2} \\ y_2 + z_2 & z_2 + x_2 & x_2 + y_2 \end{vmatrix} \right\}^q (\phi_1 \psi_2)$$

on replacing x_1, y_1, z_1; x_2, y_2, z_2 by x, y, z after the differentiations have been executed. It will, of course, be understood that the factors in each cross product of the determinants above are to be taken in *their natural order*, *i. e.*

$$\begin{vmatrix} \delta_{x_1} & \delta_{y_1} & \delta_{z_1} \\ \delta_{x_2} & \delta_{y_2} & \delta_{z_2} \\ y_1 + z_1 & z_1 + x_1 & x_1 + y_1 \end{vmatrix}^{\mu}$$ is to be understood to mean, not $[\Sigma(x_1 + y_1)(\delta_{x_1} \delta_{y_2} - \delta_{y_1} \delta_{x_2})]^{\mu}$, but $[\Sigma(\delta_{x_1} \delta_{y_2} - \delta_{y_1} \delta_{x_2})(x_1 + y_1)]^{\mu}$, and so in general.

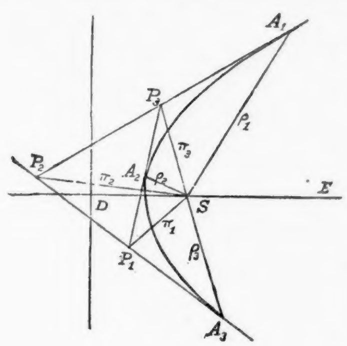
The result of this investigation has been to open my eyes to the unquestionable fact that, as we know that the first "Ueberschiebung," or "transvectant," or "alliance," of two or more quantics (names significant and useful enough to indicate the particular modes under which they are considered to be generated) is the ordinary Jacobian, so the right general name for the Ueberschiebung or alliance of any order viewed *per se* (as a *Ding an sich*) and without reference to its mode of origination, which ought to supersede all others, is the *Jacobian of the corresponding order;* or, in other words, the theory of invariants falls into the theory of compound differentiation, and just as $\left(\frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx}\right)$ is called a Jacobian and designated by $\frac{d\,(u, v)}{d\,(x, y)}$, so $\frac{d^2u}{dx^2}\frac{d^2v}{dy^2} - \frac{d^2u}{dxdy}\frac{d^2v}{dxdy} + \frac{d^2u}{dy^2}\frac{d^2v}{dx^2}$ is entitled to be called the second Jacobian and to be designated by $\frac{d^2\,(u, v)}{d\,(x, y)^2}$, and more generally every hyperdeterminant may be designated as a compound differential coefficient (or derivative) of the type $\frac{d^{\alpha}d^{\beta}\ldots}{d(\ \ )^{\alpha}d(\ \ )^{\beta}\ldots}$, where the vacant spaces are to be filled up by the insertion of a certain number of letters, with liberty for any number of them in each parenthesis to be identical with the like number in any other. Since we are now in possession of a definite analogue to ordinary differential coeffi-

cients of all orders, I do not know whether I shall be considered too bold or fanciful in suggesting that there ought to exist, in the nature of things, some theorem of development for several sets of variables analogous to Taylor's for a single set: what such theorem is or could be I have at present no conception, but as little, be it remembered, could any one, even Jacobi himself, before the creation of hyperdeterminants, have had the remotest conception in regard to a function of several variables bearing to $\left(\frac{d}{dx}\right)^i \phi$ the same relation of analogy as the ordinary functional determinant to $\frac{d\phi}{dx}$, whether such function could exist, and if so, what it would be. I have always thought and felt that beyond all others the algebraist, in his researches, needs to be guided by the principle of faith, so well and philosophically defined as "the substance of things hoped for, the evidence of things not seen."

THE TANGENT TO THE PARABOLA.

By M. L. Holman and E. A. Engler, *St. Louis, Mo.*

It is proposed in this paper to discuss, by the quaternion method, the relations between three intersecting tangents to the parabola.



Suppose tangents to be drawn at any three points as A_1, A_2, A_3; designate the vectors from the focus to these points by ρ_1, ρ_2, ρ_3 respectively. These tangents intersect each other at P_1, P_2, P_3; designate the vectors from the focus to these points of intersection by π_1, π_2, π_3 respectively.

Let α be a unit vector parallel to the axis, and β a unit vector parallel to the directrix; then the equation of the parabola referred to the focus as origin is

$$\rho = \frac{1}{4a} y^2 \alpha + y\beta - a\alpha = \frac{1}{4a}(y^2 - 4a^2)\,\alpha + y\beta\,, \tag{1}$$

in which ρ is any radius-vector and $a = \frac{1}{2} SD =$ the distance from the focus to the vertex. The vector along the tangent is $\frac{y}{2a}\alpha + \beta$.

Since ρ_1 and ρ_2 are vectors to the curve we have, by (1),

$$\rho_1 = \frac{1}{4a}(y_1^2 - 4a^2)\,\alpha + y_1\beta\,, \qquad \rho_2 = \frac{1}{4a}(y_2^2 - 4a^2)\,\alpha + y_2\beta\,;$$

from which equations, since $\rho^2 = -(\mathrm{T}\rho)^2$,

$$\mathrm{T}\rho_1\,\mathrm{T}\rho_2 = \frac{1}{16a^2}(y_1^2 + 4a^2)(y_2^2 + 4a^2)\,. \tag{2}$$

Again

$$\pi_1 = \rho_2 + A_2 P_1 = \frac{1}{4a}(y_2^2 - 4a^2)\,\alpha + y_2\beta + z\left(\frac{1}{2a} y_2\alpha + \beta\right),$$

$$\pi_1 = \rho_3 + A_3 P_1 = \frac{1}{4a}(y_3^2 - 4a^2)\,\alpha + y_3\beta + x\left(\frac{1}{2a} y_3\alpha + \beta\right),$$

in which x and z are unknown coefficients, easily determined by vector comparison, thus: $x = \frac{1}{2}(y_2 - y_3)$, $z = \frac{1}{2}(y_3 - y_2)$, and

$$\pi_1 = \frac{1}{4a}(y_2 y_3 - 4a^2)\,\alpha + \tfrac{1}{2}(y_2 + y_3)\,\beta\,, \tag{3}$$

whence, and by a cyclic change of suffices,

$$\left.\begin{aligned}(T\pi_1)^2 &= \frac{1}{16a^2}(y_2^2+4a^2)(y_3^2+4a^2)\\ (T\pi_2)^2 &= \frac{1}{16a^2}(y_3^2+4a^2)(y_1^2+4a^2)\\ (T\pi_3)^2 &= \frac{1}{16a^2}(y_1^2+4a^2)(y_2^2+4a^2)\end{aligned}\right\} \quad (4)$$

Hence, by (2) and the analogous formulae,

$$(T\pi_1)^2 = T\rho_2\, T\rho_3, \quad (T\pi_2)^2 = T\rho_3\, T\rho_1, \quad (T\pi_3)^2 = T\rho_1\, T\rho_2, \quad (5)$$

or *the distance from the focus of a parabola to the intersection of two tangents is a mean proportional between the radii-vectores to the points of contact.*

Equation (3) shows that *the distance of the point of intersection of two tangents, from the axis, is the arithmetical mean between the ordinates to their points of contact.*

From (5) we deduce

$$T\pi_2\, T\pi_3 = T\rho_1\, T\pi_1, \quad T\pi_3\, T\pi_1 = T\rho_2\, T\pi_2, \quad T\pi_1\, T\pi_2 = T\rho_3 . T\pi_3 . \quad (6)$$

By (3) and the analogous formulae

$$P_2 P_3 = \pi_3 - \pi_2 = \frac{1}{4a} y_1 (y_2 - y_3)\, \alpha + \tfrac{1}{2}(y_2 - y_3)\, \beta,$$

$$A_3 P_1 = \pi_1 - \rho_3 = \frac{1}{4a} y_3 (y_2 - y_3)\, \alpha + \tfrac{1}{2}(y_2 - y_3)\, \beta .$$

Squaring and taking the tensors of these two expressions,

$$\overline{P_2 P_3}^2 = \frac{1}{16a^2}(y_3 - y_2)^2 (y_1^2 + 4a^2), \quad \overline{A_3 P_1}^2 = \frac{1}{16a^2}(y_3 - y_2)^2 (y_3^2 + 4a^2). \quad (7)$$

Then, by (4) and (7),

$$\frac{(T\pi_3)^2}{\overline{P_2 P_3}^2} = \frac{y_2^2 + 4a^2}{(y_3 - y_2)^2}, \quad \frac{(T\pi_1)^2}{\overline{A_3 P_1}^2} = \frac{y_2^2 + 4a^2}{(y_3 - y_2)^2}, \quad (8)$$

and by (6) and (8),

$$\frac{P_2 P_3}{A_3 P_1} = \frac{T\pi_3}{T\pi_1} = \frac{T\pi_2}{T\rho_3}, \quad \frac{P_3 P_1}{A_1 P_2} = \frac{T\pi_1}{T\pi_2} = \frac{T\pi_3}{T\rho_1}, \quad \frac{P_1 P_2}{A_2 P_3} = \frac{T\pi_2}{T\pi_3} = \frac{T\pi_1}{T\rho_2}. \quad (9)$$

From these equations the following relations between angles readily appear:

$$\begin{aligned} A_1 P_2 X^* &= P_2 S A_3 = A_1 S P_2 = P_3 S P_1,\\ S A_1 P_2 &= S P_2 A_3 = S P_3 P_1,\\ S A_3 P_2 &= S P_2 A_1 = S P_1 P_3,\end{aligned}$$

all of which, when A_1 and A_3 are fixed, are constant angles. Equations (9) furnish a proof for a well-known method of constructing the parabola.

* The letter X, which through an oversight is wanting in the diagram, is any point of $A_3 P_2$ extended.

Thus far the consideration has been entirely general, the points of contact being any points on the curve. The special cases are significant as indicative of the neatness and simplicity of quaternion methods; a few will therefore be given.

1. *When* ρ_2 *is a multiple of* ρ_1, *or when* $\rho_2 - \rho_1$ *is a focal chord,* we have $x\rho_2 = \rho_1$, in which the sign of x is essentially negative.

Then by (1)

$$x\left[\frac{1}{4a}(y_2^2 - 4a^2)\,\alpha + y_2\beta\right] = \frac{1}{4a}(y_1^2 - 4a^2)\,\alpha + y_1\beta\,,$$

$$\therefore\quad x = \frac{y_1^2 - 4a^2}{y_2^2 - 4a^2} = \frac{y_1}{y_2}\,,\quad y_1(y_1y_2 + 4a^2) = y_2(y_1y_2 + 4a^2)\,,$$

which can only be satisfied, under the given conditions, by $y_1y_2 + 4a^2 = 0$.

Then, by (1) and (3),

$$S\pi_3\rho_1 = -\frac{1}{16a^2}(y_1^2 + 4a^2)(y_1y_2 + 4a^2) = 0\,,$$

or π_3 is perpendicular to ρ_1; *i. e. the line from the focus to the intersection of the tangents at the extremities of a focal chord is perpendicular to the focal chord.*

Also, by (5),

$$S(\rho_1 - \pi_3)(\rho_2 - \pi_3) = S\rho_1\rho_2 + \pi_3^2 = 0\,,$$

or *the tangents at the extremities of a focal chord are perpendicular to each other.*

Since $y_1y_2 + 4a^2 = 0$, we have

$$\pi_3 = \frac{1}{4a}(y_1y_2 - 4a^2)\,\alpha + \tfrac{1}{2}(y_1 + y_2)\,\beta = -2a\alpha + \tfrac{1}{2}(y_1 + y_2)\,\beta\,,$$

i. e. the tangents at the extremities of a focal chord meet on the directrix.

2. *When* ρ_2 *becomes a multiple of* β,

$$\rho_2 = \frac{1}{4a}(y_2^2 - 4a^2)\,\alpha + y_2\beta = x\beta\,,$$

$$x = y_2 = \pm 2a\,;$$

i. e. the parameter equals the double ordinate through the focus, or twice the distance from the focus to the directrix.

3. *When* ρ_2 *becomes a multiple of* α, $y_2 = 0$, *and*

$$\pi_3 = -a\alpha + \tfrac{1}{2}y_1\beta\,;$$

i. e. the subtangent is bisected at the vertex.

Also, $$\pi_3 - \rho_1 = -a\alpha + \frac{y_1}{2}\beta - \left(\frac{y_1^2 - 4a^2}{4a}\alpha + y_1\beta\right) = -\frac{1}{4a}y_1^2\,\alpha - \tfrac{1}{2}y_1\beta\,,$$

$$S\pi_3(\pi_3 - \rho_1) = 0\,;$$

i. e. the perpendicular from the focus on a tangent meets it on the principal tangent.

Again, the normal at A_1 may be written

$$x\pi_3 = x\,(-a\alpha + \tfrac{1}{2}y_1\beta) = z\alpha + y_1\beta, \quad x = 2; \quad z = -2a;$$

showing that *the subnormal is constant;* and that *the normal is twice the perpendicular from the focus on the tangent.*

Also, $x\pi_3 = -z\alpha + \rho_1$, $x\,(-a\alpha + \frac{1}{2}y_1\beta) = -z\alpha + \frac{1}{4a}(y_1^2 - 4a^2)\,\alpha + y_1\beta$,

$$x = 2; \quad z = \frac{1}{4a}(y_1^2 + 4a^2) = T\rho_1,$$

which shows that *the distance from the foot of the normal to the focus equals the radius vector to the point of contact, or the distance from the foot of the tangent to the focus, or the distance from the point of contact to the directrix;* proofs for each of which may also be obtained directly by writing the proper vector equalities as obtained from the figure and determining the value of the unknown coefficients as above.

The part of the tangent from its foot to the point of contact is readily found to be $\frac{1}{2a}y_1^2\alpha + y_1\beta$, since it is equal to $z\alpha + \rho_1$, and z has the value $\frac{1}{4a}(y_1^2 + 4a^2)$, as already shown in our last equation. The part of the tangent from the foot of the focal perpendicular to the point of contact is

$$-\pi_3 + \rho_1 = \frac{1}{4a}y_1^2\alpha + \tfrac{1}{2}y_1\beta,$$

i. e. the tangent is bisected by the focal perpendicular. Whence *the angle between the tangent and the radius vector equals the angle between the tangent and the axis, or the angle between the normal and the directrix;* and also, *the tangent bisects the angle between the diameter and the focal radius to the point of contact.*

The perpendicular from the focus on the normal is also $-\pi_3 + \rho_1$, whose expression just given proves that *the locus of the foot of the perpendicular from the focus on the normal is a parabola, whose vertex is at the focus of the given parabola, and whose parameter is $\frac{1}{4}$ of that of the given parabola.*

The vector to the middle point of a focal chord (when ρ_2 is a multiple of ρ_1) is

$$\rho = \tfrac{1}{2}(\rho_1 + \rho_2) = \tfrac{1}{2}\left(1 + \frac{y_2}{y_1}\right)\rho_1 = \frac{1}{2y_1^2}(y_1^2 - 4a^2)\,\rho_1$$

$$= \frac{1}{8ay_1^2}(y_1^2 - 4a^2)^2\,\alpha + \frac{1}{2y_1}(y_1^2 - 4a^2)\,\beta,$$

that is, *the locus of the middle points of focal chords is a parabola, whose vertex is at the focus of the given parabola, and whose parameter is the parameter of the given parabola.*

4. *When π_3 becomes a multiple of β;*

$$\pi_3 = \frac{1}{4a}(y_1y_2 - 4a^2)\,\alpha + \tfrac{1}{2}(y_1 + y_2)\,\beta = x\beta$$

$$\therefore\ y_1y_2 = 4a^2,\ x = \tfrac{1}{2}(y_1 + y_2) = \frac{1}{2y_1}(y_1^2 + 4a^2)\,,$$

$$\pi_3 = \frac{1}{2y_1}(y_1^2 + 4a^2)\,\beta\,,\ \ \pi_3^2 = -\frac{1}{4y_1^2}(y_1^2 + 4a^2)^2.$$

When the point P_3 coincides with the intersection of the tangent A_1P_3 with the directrix, we have

$$\pi_3 = \frac{y_1y_2 - 4a^2}{4a}\,\alpha + \frac{y_1 + y_2}{2}\,\beta = -2a\alpha + x\beta\,,$$

$$\therefore\ y_1y_2 = -4a^2,\ x = \frac{y_1 + y_2}{2} = \frac{1}{2y_1}(y_1^2 - 4a^2)\,,$$

$$\pi_3 = -2a\alpha + \frac{1}{2y_1}(y_1^2 - 4a^2)\beta\,,\ \ \pi_3^2 = -\frac{1}{4y_1^2}(y_1^2 + 4a^2)^2,$$

hence, *any tangent intersects the latus rectum and directrix in points equally distant from the focus.*

ADDENDA TO MR. HALSTED'S PAPER ON THE BIBLIOGRAPHY OF HYPER-SPACE AND NON-EUCLIDEAN GEOMETRY.

(Pages 261–276.)

Titles marked * were furnished by Mr. Halsted; those marked † were given in the Academy, October 26th; the others have been suggested by various individuals.

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(Further Addenda will be given in the next Volume.)

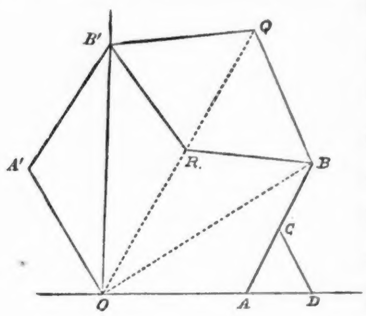
NOTES.

I.

Link-Work for x^2.

(*Extract from a Letter of* PROFESSOR CAYLEY *to* MR. SYLVESTER.)

I suppose the following is substantially your link-work for x^2. I use a slot to make D move in the line OA; but this could be replaced by proper link-work. Supposing O and A fixed; the line OB is movable, and I wanted to have the distance OB measured in a fixed direction. This can be done by a hexagon $OABQB'A'$ with equal sides, and two other equal links $B'R$, BR: then of course, if O, R, Q are in lineâ, the hexagon will be symmetrical as to OQ, and OB' will be equal OB, and B' may be made to move in the fixed line OB'.



$$BOA = \tfrac{1}{2}\theta, \quad OA = AB = a, \quad AC = CD = \tfrac{1}{2}a,$$

then $$OB = 2a\cos\tfrac{1}{2}\theta, \quad OD = a(1+\cos\theta) = 2a\cos^2\tfrac{1}{2}\theta,$$

or $$2a\,.\,OD = (OB)^2.$$

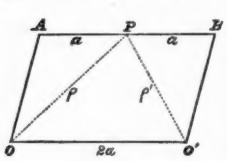
30TH NOVEMBER, 1877.

II.

Link-Work for the Lemniscate.

(*Extract from a Letter of* MR. A. W. PHILLIPS *of New Haven.*)

IN regard to the description of the Lemniscate I found in working upon the 3-bar link-work, if O and O' were fixed points, $ABO'O$ a parallelogram, $AP = PB$ and $OA = r = \sqrt{2}a$, that P would describe the lemniscate of which O and O' were the foci. The complete equation of the curve described by P expressed in bipolar coordinates reduces to



$$(\rho^2 + \rho'^2 - 6a^2)(\rho^2\rho'^2 - a^4) = 0,$$

where $\rho^2 + \rho'^2 - 6a^2 = 0$ is the equation of a circle in bipolars. Then $\rho^2\rho'^2 - a^4 = 0$ gives $\rho\rho' = a^2$ the equation to the lemniscate.

8TH MAY, 1878.

III.

Euler's Equations of Motion.

BY JAMES LOUDON, *University College, Toronto, Canada.*

1. A rigid body fixed at O has at time t rotations, ω_1, ω_2, ω_3 round the principal axes OA, OB, OC, to determine the changes per unit time in these rotations.

The positions OA', OB', OC' of the axes at time $t+\delta t$ will be known from the displacements in time δt, due to these rotations, of the points $A(\omega_1, 0, 0)$, $B(0, \omega_2, 0)$, $C(0, 0, \omega_3)$. The components of these displacements in the directions OA, OB, OC, respectively, are evidently

$$\begin{array}{cccl} 0, & \omega_1\omega_3\delta t, & -\omega_1\omega_2\delta t, & \text{for } A \\ -\omega_2\omega_3\delta t, & 0, & \omega_2\omega_1\delta t, & \text{for } B \\ \omega_3\omega_2\delta t, & -\omega_3\omega_1\delta t, & 0, & \text{for } C. \end{array}$$

The component rotations at time $t+\delta t$ are $\omega_1 + \frac{d\omega_1}{dt}\delta t$, &c., which may be represented by OA', OB', OC'. The changes of the rotations in time δt are, therefore, AA', BB', CC'. Resolving these changes into the components (AF, FP, PA'), (BG, GQ, QB'), (CH, HR, RC') in the directions of the axes at time t, we get (observing that FP, PA' are the displacements in time δt of the point $F\left(\omega_1 + \frac{d\omega_1}{dt}\delta t, 0, 0\right)$, &c., and neglecting infinitesimals of the first order) the following as the resultant changes in time δt:

$$AF + GQ + HR = \left(\frac{d\omega_1}{dt} - \omega_2\omega_3 + \omega_3\omega_2\right)\delta t = \frac{d\omega_1}{dt}\delta t \text{ along } OA$$

$$FP + BG + RC' = \left(\omega_1\omega_3 + \frac{d\omega_2}{dt} - \omega_3\omega_1\right)\delta t = \frac{d\omega_2}{dt}\delta t \text{ along } OB$$

$$PA' + QB' + CH = \left(-\omega_1\omega_2 + \omega_2\omega_1 + \frac{d\omega_3}{dt}\right)\delta t = \frac{d\omega_3}{dt}\delta t \text{ along } OC.$$

The changes per unit time are therefore $\frac{d\omega_1}{dt}$, $\frac{d\omega_2}{dt}$, $\frac{d\omega_3}{dt}$ in the directions OA, OB, OC, respectively.

2. To determine the component changes of the body's moment of momentum. At time t the components of the moment of momentum are $A\omega_1$, $B\omega_2$, $C\omega_3$ in the directions of the principal axes, (where A, B, C denote the principal moments of inertia. At time $t+\delta t$ the components are $A\left(\omega_1 + \frac{d\omega_1}{dt}\delta t\right)$, &c., in the directions OA', OB', OC'. Employing the figure in a new sense, the former components may be represented by OA, OB, OC and the latter by OA', OB', OC'. The changes of the moment of momentum in time δt are therefore AA', BB', CC'. Resolving these changes into their components parallel to the axes at time t we get, as in the former case, (observ-

ing that FP, PA' are now the displacements in time δt of the point F, $\left(A\omega_1 + A\frac{d\omega_1}{dt}\delta t,\ 0,\ 0\right)$, &c.), the following as the resultant changes of the moment of momentum in time δt:

$$\left(A\frac{d\omega_1}{dt} - B\omega_2\omega_3 + C\omega_3\omega_2\right)\delta t \text{ along } OA$$

$$\left(A\omega_1\omega_3 + B\frac{d\omega_2}{dt} - C\omega_3\omega_1\right)\delta t \text{ along } OB$$

$$\left(-A\omega_1\omega_2 + B\omega_2\omega_1 + C\frac{d\omega_3}{dt}\right)\delta t \text{ along } OC.$$

The changes per unit time are therefore $A\frac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3$, &c., in the directions OA, OB, OC, respectively.

21st November, 1878.

IV.

Condition of a Straight Line Touching a Surface.

By James Loudon, *University College, Toronto, Canada.*

The following method of getting the condition of a straight line touching a surface I have used for some years, and expected to see it in Salmon's 3d edition. It was so short and unimportant that I never thought of publishing it. (See Salmon's *Geometry of Three Dimensions, Art.* 80.)

Let $u \equiv ax^2 + by^2 + cz^2 + dw^2 + 2fyz + 2gzx + 2hxy + 2lxw + 2myw + 2nzw = 0$, then $P \equiv \alpha x + \beta y + \gamma z + \delta w = 0$, $P' \equiv \alpha' x + \beta' y + \gamma' z + \delta' w = 0$ is a line touching $u = 0$. Therefore $P - \lambda P' = 0$ is, for some value of λ, the tangent plane at the point of contact (x', y', z', w'); *i. e.*

$(\alpha - \lambda\alpha')\,x + (\beta - \lambda\beta')\,y + (\gamma - \lambda\gamma')\,z + (\delta - \lambda\delta')\,w = 0$ and
$(ax' + hy' + gz' + lw')\,x + (hx' + by' + fz' + mw')\,y + (gx' + fy' + cz' + nw')\,z + (lx' + my' + nz' + dw')\,w = 0$ are identical. Therefore

$$\begin{array}{ll} ax' + hy' + gz' + lw' = k\,(\alpha - \lambda\alpha') & hx' + by' + fz' + mw' = k\,(\beta - \lambda\beta') \\ gx' + fy' + cz' + nw' = k\,(\gamma - \lambda\gamma') & lx' + my' + nz' + dw' = k\,(\delta - \lambda\delta') \\ \alpha x' + \beta y' + \gamma z' + \delta w' = 0 & \alpha' x' + \beta' y' + \gamma' z' + \delta' w' = 0. \end{array}$$

Eliminating $x', y', z', w', -k, k\lambda$, we get the required condition

$$\begin{vmatrix} a, & h, & g, & l, & \alpha, & \alpha' \\ h, & b, & f, & m, & \beta, & \beta' \\ g, & f, & c, & n, & \gamma, & \gamma' \\ l, & m, & n, & d, & \delta, & \delta' \\ \alpha, & \beta, & \gamma, & \delta, & 0, & 0 \\ \alpha', & \beta', & \gamma', & \delta', & 0, & 0 \end{vmatrix} = 0.$$

28th November, 1878.